HIGHER F-SINGULARITIES FOR CONES

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Abstract. We provide geometric criteria for higher F-injectivity and F-rationality of affine cones in characteristic p. If X is a smooth projective variety and $\mathcal{L}$ is an ample line bundle on X, the cone $C = C(X, L)$ is singular at the vertex. For each $k \in \mathbb{N}$, we relate the vertex singularity being (k, F)-injective or (k, F)-rational to certain ordinarity properties on X and cohomological vanishings for sheaves twisted by L. We compare the ordinarity properties of X to other common notions of ordinarity such as Bloch–Kato ordinary varieties. As an application, we connect our findings to the study of Kuznetsov components of smooth cubic fourfolds.

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1. Introduction

A fundamental problem in singularity theory is to relate the geometry of a projective variety X equipped with an ample line bundle L to the singularities of its affine cone with conormal bundle L:

$$C(X, L) := \operatorname{Spec}\left(\bigoplus_{m \geq 0} \mathrm{H}^0(X, L^m)\right).$$

As an example, if X is defined over a field of characteristic 0, we may relate the singularities of the cone to the positivity of the canonical divisor K_X of X: assuming X is smooth projective and $C(X, L)$ is $\mathbb{Q}$-Gorenstein, we have that $C(X, L)$ is Kawamata log-terminal if and only if $K_X < 0$ (i.e. X is a Fano variety). Similarly $C(X, L)$ is log canonical if $K_X \sim 0$

(i.e. X is Calabi–Yau) and $C(X, L)$ is not log canonical if and only if $K_X > 0$ (i.e. X is a variety of general type).

In the study of characteristic p cone singularities, one also obtains a link to arithmetic (rather than just geometric) properties of the underlying smooth variety X. As an example, the characteristic p analogue of log canonical is *F-split*, i.e. the Frobenius $F : \mathcal{O}_{C(X,L)} \to F_*\mathcal{O}_{C(X,L)}$ splits in the category of $\mathcal{O}_{C(X,L)}$-modules, which is equivalent to X being globally F-split. In this context, Mehta and Srinivas proved the following:

Theorem 1.1 (Mehta–Srinivas [MS87a])**.** *Let X be a smooth Calabi–Yau variety of dimension d over an algebraically closed field of characteristic p. Then $C(X, L)$ is F-split if and only if X is weakly ordinary in the sense that $F : \mathrm{H}^d(X, \mathcal{O}_X) \to \mathrm{H}^d(X, \mathcal{O}_X)$ is injective.*

In this paper, we explore deeper connections between the arithmetic and geometric properties of X, and certain F-singularity properties of the associated cone $C(X, L)$, in the spirit of Mehta–Srinivas. The "arithmetic" here consists of more general ordinarity conditions on X, and the F-singularities we are interested in are *higher F-injective and F-rational* singularities as developed in [KW24], [KW26]. We recall their definition here:

Definition 1.2 (See Definitions 2.8, 2.11)**.** *Suppose X is a normal d-dimensional variety over a characteristic p field with an isolated singularity $\mathfrak{m} \in X$. Let $B\Omega^\bullet_X \subseteq F_*\Omega^\bullet_X$ and $B_n\Omega^\bullet_X \subseteq F^n_*\Omega^\bullet_X$ denote the coboundaries and higher coboundaries of the Frobenius-twisted de Rham complex, and let $Z\Omega^\bullet_X \subseteq F_*\Omega^\bullet_X$ denote the cocycles. Write $\Omega^{[i]}_X, B_n\Omega^{[i]}_X$ and $Z\Omega^{[i]}_X$ for their reflexivizations. Suppose for a fixed k that the* Cartier map $C : Z\Omega^{[i]}_X \to \Omega^{[i]}_X$ *is surjective for $i \leq k$. We say the singularity $(X, \mathfrak{m})$ is*

(i) pre-(k, F)-injective *if the natural inverse Cartier map*

$$C^{-1} : \mathrm{H}^j_{\mathfrak{m}}\left(\Omega^{[i]}_X\right) \to \mathrm{H}^j_{\mathfrak{m}}\left(\frac{F_*\Omega^{[i]}_X}{B\Omega^{[i]}_X}\right)$$

is injective for $i \leq k$, $i + j \leq d$. The singularity is (k, F)-injective *if in addition Ω^i_X is reflexive for $i \leq k$.*

(ii) pre-(k, F)-rational *if for every Cartier divisor $D \geq 0$ on X, the composite map*

$$\mathrm{H}^j_{\mathfrak{m}}\left(\Omega^{[i]}_X\right) \xrightarrow{C^{-1}_n} \mathrm{H}^j_{\mathfrak{m}}\left(\frac{F^n_*\Omega^{[i]}_X}{B_n\Omega^{[i]}_X}\right) \longrightarrow \mathrm{H}^j_{\mathfrak{m}}\left(\frac{F^n_*\Omega^{[i]}_X(D)}{B_n\Omega^{[i]}_X}\right)$$

is injective for $n \gg 0$, $i \leq k$ and $i + j \leq d$. The singularity is (k, F)-rational *if in addition $k \leq d/2 - 1$.*

We refer to Section 2 for the constructions of Cartier maps C, C_n and $B_n\Omega$, as well as for more general definitions and examples of higher F-injective and F-rational singularities.

Specializing to $k = 0$ recovers the more familiar notion of F-injective and F-rational singularities; in particular, F-split singularities are F-injective.

The characteristic zero analogues of higher F-injective and F-rational singularities are *k-Du Bois* and *k-rational* singularities respectively, which are defined through the Du Bois complex and have been extensively studied in [FL22], [FL24], [MOPW23], [CDM24] and [SVV23] via Hodge-theoretic methods. Inspired by the work of [Sch09] and [BST17], Kawakami–Witaszek have linked these characteristic zero and characteristic p singularities via reduction modulo p: more precisely, a normal affine variety $X/\mathbb{C}$ with isolated singularities and Ω^i_X

reflexive for $i \leq k$ fixed has k-Du Bois singularities if and only if it admits a spreadout $\mathcal{X}$ over a $\mathbb{Z}$-algebra so that there is a Zariski dense set of mod p (k,F)-injective fibers $\mathcal{X}_{p=0}$ (see [KW24, Theorem A]– the "only if" case contingent on the ordinarity conjecture. In a similar way they have also shown that X has k-rational singularities if and only if $\mathcal{X}_{p=0}$ has (k,F)-rational singularities for a Zariski dense set of p (see [KW26, Theorem C]); this result is interestingly not contingent on the weak ordinarity conjecture.

The higher analogue of ordinarity we will study is the following.

Definition 1.3. *A smooth projective d-dimensional variety X is called* skew k-ordinary *if* $\mathrm{H}^j(X, B\Omega^i_X) = 0$ *for $i \leq k$ and $i + j \leq d$.*

Skew k-ordinary varieties are weakened forms of k-ordinary varieties in the sense of Bloch–Kato (see [BK86, §7]). When $k = 1$, skew and Bloch–Kato 1-ordinary coincide and are equivalent to $F : \mathrm{H}^j(X, \mathcal{O}_X) \to \mathrm{H}^j(X, \mathcal{O}_X)$ being bijective for all j; in particular for $j = \dim(X)$ we recover weak ordinarity. We refer to Section 3 for more on k-ordinary varieties.

Our first theorem connects the (k,F)-injectivity of the cone to an ordinarity property of X and some cohomological vanishing which may be interpreted as a weak version of Bott vanishing in characteristic p.

Theorem A (Theorem 5.9, Corollary 5.7.1, Proposition 5.11)**.** *Suppose that X is a smooth projective variety of dimension $d \geq 2$ over a perfect field of characteristic $p > 0$ and let L be an ample line bundle on X with $c_1(L) = 0$. For a fixed $k \leq d-2$, the affine cone $C(X,L)$ is pre-(k,F)-injective if and only if:*

(i) *X is skew $(k+1)$-ordinary.*
(ii) *$\mathrm{H}^j(X, \Omega^i_X \otimes L^m) = 0$ for $m \neq 0$, $i \leq k$ and $1 \leq j < d - i$.*
(iii) *$\mathrm{H}^{d-i-1}(X, B\Omega^{i+1}_X \otimes L^m) = 0$ for $m \neq 0$ and $i \leq k$.*

If $d = 1$ (i.e. X is a curve) then $C(X,L)$ is pre-(k,F)-injective for any k if and only if it is F-injective, and $C(X,L)$ is (k,F)-injective if and only if it is smooth.

For an arbitrary ample L (i.e. without assuming $c_1(L) = 0$), we get a criterion for pre-(k,F)-injectivity of the cone by keeping (i) and replacing (ii) and (iii) with the cohomological vanishing of certain "weight zero" sheaves of differentials pushed forward along the natural $\mathbb{G}_m$-torsor $\pi : C(X,L)\setminus\{v\} \to X$; see Corollary 5.7.1 for details. This extends to a (k,F)-injectivity criterion if we furthermore assume $\Omega^i_{C(X,L)}$ are reflexive for all $i \leq k$.

Condition (ii) appears in the relevant characteristic zero criterion for k-Du Bois singularities of cones (see [SVV23, §7]), but only for $m > 0$; the case $m < 0$ is automatic by Kodaira–Akizuki–Nakano vanishing. Conditions (i) and (iii) are unique to characteristic p; in particular, (iii) has interest even for $k = 0$, where it may be reinterpreted in terms of Frobenius-stable sections of $\omega_X \otimes L^m$; see Section 5 for details.

We also note that even when X is ordinary, the cohomological conditions (ii)-(iii) need not be satisfied, and so the cone may fail to be higher F-injective – for example an ordinary curve X of genus ≥ 2 with $L = \omega_X$ has a non-F-injective cone. However, choosing a sufficiently ample polarization L makes (ii)-(iii) automatic by Serre vanishing, and Theorem A establishes a direct relationship between higher (pre-) F-injectivity of the cone and ordinarity of its base smooth variety; see Corollary 5.11.1.

The next natural question is the existence of a (k,F)-rationality criterion for cones in terms of properties intrinsic to X and L. Our second theorem answers this question:

Theorem B (Proposition 6.5, Corollary 6.5.1)**.** *Suppose X is a smooth projective variety of dimension d over a perfect field of characteristic p, which furthermore satisfies Hodge symmetry. Let L be an ample line bundle on X and fix $1 \leq k \leq d-2$. The cone $C(X,L)$ is pre-(k,F)-rational if and only if:*

(i) *$C(X,L)$ is pre-(k,F)-injective.*
(ii) *$\mathrm{H}^j(X,\Omega_X^i) = 0$ for $i \leq k$ and $1 \leq j \neq i$.*
(iii) *The cup product with $c_1(L)$ induces isomorphisms*

$$\mathrm{H}^{i-1}(X,\Omega_X^{i-1}) \xrightarrow{\sim} \mathrm{H}^i(X,\Omega_X^i), \qquad 2 \leq i \leq k$$

and is surjective for $i = 1$.

If $k < d/2$, $C(X,L)$ is (k,F)-rational if and only if (i)-(iii) *above hold.*

This gives a characteristic p counterpart to the characteristic zero k-rationality criterion for cones as established in [SVV23, §7].

We observe the following statements about Fano varieties of K3 type as corollaries: here an n-dimensional Fano variety of K3 type satisfies $\mathrm{H}^i(X,\Omega_X^{n-i}) = 0$ for $i < k$, $\dim \mathrm{H}^k(X,\Omega_X^{n-k}) = 1$ for a fixed k.

Theorem C (Theorem A, Corollary 3.8.1)**.** *Let X be a Fano variety of K3 type of dimension n, then the following are equivalent:*

(i) *X is ordinary.*
(ii) *X is Bloch–Kato $(k+1)$-ordinary.*
(iii) *For some sufficiently ample line bundle L, $C(X,L)$ is pre-(k,F)-injective.*

In Appendix A we discuss the particular case of cubic fourfolds, which are Fano of K3 type (with $k = 1$). When the Kuznetsov component of X is naturally identified with the bounded derived category of a (twisted) K3 surface Y, we have that X is ordinary if and only if Y is ordinary, if and only if the cone $C(Y,L)$ is F-injective for some ample line bundle L on Y.

Outline. Section 2 reviews the behavior of Cartier operators on singular normal varieties and defines the main classes of singularities we study, namely higher F-injective and higher F-rational singularities. Section 3 deals with the different notions of k-ordinary varieties in positive characteristic that are relevant for our purposes. In Section 4 we construct the affine cone and study the associated residue exact sequences of higher differential sheaves over cones and their underlying smooth varieties. These technical preliminaries are then used in Sections 5 and 6 to prove Theorem A and Theorem B respectively. Finally, in Appendix A we relate the k-ordinarity of cubic fourfolds and K3 surfaces via Kuznetsov components.

Conventions and notations.

- All varieties by definition will be integral, separated and of finite type over a field.
- If k is a field of characteristic p and R is a k-algebra, we denote by ϕ the Frobenius on k and by $F : R \to F_*R$ the Frobenius on R, i.e. the p-th power map $a \mapsto a^p$. Here F_*R is R equipped with R-module structure $r \cdot a = r^p a$. This makes F into an R-module homomorphism, but note it is a ϕ-semilinear k-algebra homomorphism.
- For any variety X we denote by $F : X \to X$ its *absolute* Frobenius, which is an identity on the underlying topological space and the p-th power map $\mathcal{O}_X \to F_*\mathcal{O}_X$ on sections. This map coincides with the Frobenius on the point above when $X = \mathrm{Spec}(R)$; in particular ϕ is the absolute Frobenius on $\mathrm{Spec}(k)$.

- For a d-dimensional variety X with dualizing sheaf ω_X, Grothendieck duality is denoted by $\mathbb{D}_X(-) = \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(-, \omega_X)$.

2. Preliminaries

2.1. **Cartier operators on singular varieties.** Let X be a variety over a perfect characteristic p field. We will provide a brief overview of Cartier operators, adapted from [KW24, §2]. We first construct the Cartier operator in the affine setting $X = \mathrm{Spec}(A)$.

Let Ω^1_A be the module of Kähler differentials with associated differential map $d : A \to \Omega^1_A$. The differential is generally not A-linear, but since $d(a^p b) = a^p db$, we can consider the complex of finitely-generated A-modules $(F_*\Omega^\bullet_A, F_*d)$ so that the differential maps

$$F_*d : F_*\Omega^i_A \to F_*\Omega^{i+1}_A$$

are A-linear. We also consider the cocycles and coboundaries of this complex, which are the following A-submodules of $F_*\Omega^i_A$:

$$Z\Omega^i_A := \ker\left(F_*d : F_*\Omega^i_A \to F_*\Omega^{i+1}_A\right)$$
$$B\Omega^i_A := \mathrm{im}\left(F_*d : F_*\Omega^{i-1}_A \to F_*\Omega^i_A\right)$$

where we set $B\Omega^0_A = 0$. We denote the cohomology sheaves of $F_*\Omega^\bullet_A$ by $\mathcal{H}^i(F_*\Omega^\bullet_A) = \frac{Z\Omega^i_A}{B\Omega^i_A}$.

To construct the Cartier operator and its global variants, we may first define a map $\gamma : A \to F_*\Omega^1_A$ assigning $\gamma(a) = F_*(a^{p-1}da)$. Via [BK05, Lemma 1.3.3], we have that

- $\gamma(ab) = a\gamma(b) + b\gamma(a)$.
- $F_*d(\gamma(a)) = 0$
- $\gamma(a+b) - \gamma(a) - \gamma(b) \in B\Omega^1_A$

Therefore, quotienting by $B\Omega^1_A$ we get an induced k-derivation $\overline{\gamma} : A \to \mathcal{H}^1(F_*\Omega^\bullet_A)$, hence by the universal property of Kähler differentials we obtain a unique A-linear map

$$C^{-1} : \Omega^1_A \to \mathcal{H}^1(F_*\Omega^\bullet_A)$$

assigning $C^{-1}(ad\eta) = [F_*(a^p\eta^{p-1}d\eta)]$. Given that $C^{-1}(d\eta) \wedge C^{-1}(d\eta) = 0$, we have that C^{-1} further extends to A-linear maps for each i:

$$C^{-1} : \Omega^i_A \to \mathcal{H}^i(F_*\Omega^\bullet_A) := \frac{Z\Omega^i_A}{B\Omega^i_A}$$

We extend the definition in degree zero via $C^{-1}\colon A \to \mathcal{H}^0(F_*\Omega^\bullet_A)$, $a \mapsto F_*(a^p)$ and, abusing notation, we refer to these maps collectively as C^{-1}. The classical Cartier isomorphism then takes the following form:

Lemma 2.1 ([BK05], Theorem 1.3.4)**.** *If A is a regular ring, then $C^{-1} : \Omega^i_A \to \frac{Z\Omega^i_A}{B\Omega^i_A}$ is an A-linear isomorphism.*

The constructions above globalize: for X any variety over a perfect characteristic p field, one can similarly define $F_*\Omega^\bullet_X$ as a complex of coherent $\mathcal{O}_X$-modules with $\mathcal{O}_X$-linear differentials $F_*d : F_*\Omega^i_X \to F_*\Omega^{i+1}_X$. The cocycles, coboundaries and Cartier operators in this setting are denoted by

$$Z\Omega^i_X := \ker\left(F_*d : F_*\Omega^i_X \to F_*\Omega^{i+1}_X\right) \in \mathrm{Coh}(X) \tag{1}$$

$$B\Omega^i_X := \mathrm{im}\left(F_*d : F_*\Omega^{i-1}_X \to F_*\Omega^i_X\right) \in \mathrm{Coh}(X) \tag{2}$$

$$C^{-1} : \Omega^i_X \to \frac{Z\Omega^i_X}{B\Omega^i_X} \ (\mathcal{O}_X\text{-linear}) \tag{3}$$

When X is regular, C^{-1} is an isomorphism by the global analogue of Lemma 2.1, and so we have a short exact sequence

$$0 \to B\Omega^i_X \to Z\Omega^i_X \xrightarrow{C} \Omega^i_X \to 0 \tag{4}$$

We record the following lemma for our purposes.

Lemma 2.2. *If X is regular and $i \geq 1$, coherent sheaves $Z\Omega^i_X$ and $B\Omega^i_X$ are locally free.*

Proof. If $d = \dim(X)$, then via the Cartier isomorphism $B\Omega^d_X \simeq \ker(\mathrm{Tr}_F : F_*\omega_X \to \omega_X)$ is a locally free $\mathcal{O}_X$-module, as it is the kernel of a surjective morphism of finitely generated locally free sheaves. Then using exact sequences

$$0 \to Z\Omega^i_X \to F_*\Omega^i_X \to B\Omega^{i+1}_X \to 0 \tag{5}$$

$$0 \to B\Omega^i_X \to Z\Omega^i_X \xrightarrow{C} \Omega^i_X \to 0 \tag{6}$$

the same argument yields successively that $Z\Omega^i_X, B\Omega^i_X$ are locally free for $i = d-1, \cdots, 1$. □

When X is not regular but still normal, we have an analogue of (4) by considering reflexivizations as follows. Let $j : U \hookrightarrow X$ be the inclusion of the (open) regular locus of X and set

$$\Omega^{[i]}_X = j_*\Omega^i_U,\ Z\Omega^{[i]}_X = j_*Z\Omega^i_U,\ \ B\Omega^{[i]}_X = j_*B\Omega^i_U$$

These sheaves agree with the usual reflexivization $\mathcal{F}^{\vee\vee}$ of a sheaf $\mathcal{F}$, where $\mathcal{F}^\vee = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$; one can see this from the normality of X (specifically from $\mathrm{codim}_X(X \setminus U) \geq 2$) and the fact that $\Omega^i_U, Z\Omega^i_U, B\Omega^i_U$ are locally free by Lemma 2.2.

By left exactness of j_*, we obtain from (4) an exact sequence

$$0 \to B\Omega^{[i]}_X \to Z\Omega^{[i]}_X \xrightarrow{C} \Omega^{[i]}_X \tag{7}$$

where $C = j_*C_U$ denotes the 'reflexivization' of the Cartier operator by pushing forward the Cartier isomorphism from U. In general (7) is not exact on the right; in special cases it is, as outlined below.

Lemma 2.3 ([KW24], Proposition 2.4)**.** *Suppose X is normal and sheaves Ω^i_X are reflexive for all $i \leq k$. Then for $i \leq k$:*

(i) *$Z\Omega^i_X, B\Omega^i_X$ are torsionfree, and moreover $Z\Omega^i_X$ is reflexive for $i \leq k-1$.*

(ii) *The reflexivized Cartier operator $C : Z\Omega^{[i]}_X \to \Omega^{[i]}_X$ is surjective, yielding an $\mathcal{O}_X$-linear isomorphism*

$$C^{-1} : \Omega^{[i]}_X \xrightarrow{\sim} \frac{Z\Omega^{[i]}_X}{B\Omega^{[i]}_X}$$

(iii) *We have $\frac{Z\Omega^i_X}{B\Omega^i_X} \simeq \frac{Z\Omega^{[i]}_X}{B\Omega^{[i]}_X} \oplus T$, where T is a torsion $\mathcal{O}_X$-module.*

The surjectivity of the reflexivized Cartier operator is more generally related to the logarithmic extension property of differential forms; see [KW26, Theorem 2.7] for some consequences.

Remark 2.4. When X is a local complete intersection with isolated singularities, the condition of Ω^i_X being reflexive for $i \leq k$ is not so restrictive. Indeed, by the Auslander–Buchsbaum formula one can deduce that Ω^i_X are reflexive for $i \leq d-2$, while Ω^{d-1}_X is torsionfree. Beyond the class of lci singularities however there are varieties where *none* of $\Omega^1_X, \cdots, \Omega^d_X$ are reflexive: the affine cone $C(\mathbb{P}^{d-1}, O(n))$ over a Veronese variety is such an example for $n > d$ and $d \geq 2$ (see Section 4.1).

In this paragraph we also make a note of *higher* Cartier operators. Assume first that X is regular and set $B_1\Omega^i_X = B\Omega^i_X, Z_1\Omega^i_X = Z\Omega^i_X$ and $C_1 = C$. We inductively define

$$Z_n\Omega^i_X = (F^{n-1}_* C)^{-1}(Z_{n-1}\Omega^i_X), \;\; B_n\Omega^i_X = (F^{n-1}_* C)^{-1}(B_{n-1}\Omega^i_X) \tag{8}$$

$$C_n = C_{n-1} \circ (F^{n-1}_* C)|_{Z_n\Omega^i_X} \tag{9}$$

where $F^{n-1}_* C : F^{n-1}_* Z\Omega^i_X \to F^{n-1}_* \Omega^i_X$. Similar to Lemma 2.2, we have:

Lemma 2.5 ([KTT+25], §5.2)**.** *When X is regular, the following hold.*

(i) *$B_n\Omega^i_X$, $Z_n\Omega^i_X$ are locally free $\mathcal{O}_X$-submodules of $F^n_*\Omega^i_X$ for all $n \geq 1$.*
(ii) *We have exact sequence $0 \to B_n\Omega^i_X \to Z_n\Omega^i_X \xrightarrow{C_n} \Omega^i_X \to 0$.*
(iii) *We have exact sequence $0 \to F^{n-1}_* B\Omega^i_X \to B_n\Omega^i_X \to B_{n-1}\Omega^i_X \to 0$ for $n \geq 2$.*

When X is normal but possibly non-regular, we once again consider the inclusion $j : U \hookrightarrow X$ of the open regular locus and set reflexivizations

$$Z_n\Omega^{[i]}_X = j_* Z_n\Omega^i_U, \;\; B_n\Omega^{[i]}_X = j_* B_n\Omega^i_U$$

We thus obtain a reflexivized higher Cartier operator $C_n : Z_n\Omega^{[i]}_X \to \Omega^{[i]}_X$, which is in general not surjective. However:

Lemma 2.6 ([KS25], Lemma 2.7)**.** *$C_n : Z_n\Omega^{[i]}_X \to \Omega^{[i]}_X$ is surjective for $n \geq 1$ if and only if $C : Z\Omega^{[i]}_X \to \Omega^{[i]}_X$ is surjective.*

Thus, in the setting where the usual reflexivized Cartier operator is surjective, we retrieve isomorphisms $C^{-1}_n : \Omega^{[i]}_X \xrightarrow{\sim} \frac{Z_n\Omega^{[i]}_X}{B_n\Omega^{[i]}_X}$ for all $n \geq 1$.

We end this subsection with a recollection of duality for cocycles and coboundaries.

Proposition 2.7 ([KW24], Lemma 2.8)**.** *When X is regular and d-dimensional, we have*

$$\mathbb{D}(F_*\Omega^i_X) \simeq F_*\Omega^{d-i}_X, \;\; \mathbb{D}(B\Omega^i_X) \simeq B\Omega^{d+1-i}_X, \;\; \mathbb{D}(Z\Omega^i_X) \simeq F_*\Omega^{d-i}_X / B\Omega^{d-i}_X$$

2.2. **(k,F)-injectivity and (k,F)-rationality.** We retain the notation and assumptions of the previous section, namely $X = \mathrm{Spec}(R)$ is an affine normal variety over a perfect field k of characteristic p. Via the reflexivized Cartier isomorphism (Lemma 2.3), when Ω^i_X are reflexive for $i \leq k$ we may write

$$C^{-1} : \Omega^{[i]}_R \cong \frac{Z\Omega^{[i]}_R}{B\Omega^{[i]}_R} \hookrightarrow \frac{F_*\Omega^{[i]}_R}{B\Omega^{[i]}_R}, \quad i \leq k$$

Note that under the assumptions of Lemma 2.3, $F_*\Omega^{[i]}_R / B\Omega^{[i]}_R$ is torsionfree. We may therefore define (k,F)-injectivity as follows.

Definition 2.8 ([KW24], §1)**.** *Let R be a normal d-dimensional domain of finite type over a perfect field k of characteristic $p > 0$. We say R is* (k,F)-injective *if*

(i) Ω_R^i *are reflexive for all* $i \leq k$
(ii)

$$C^{-1} : \mathrm{H}^j_{\mathfrak{m}}\left(\Omega_R^{[i]}\right) \to \mathrm{H}^j_{\mathfrak{m}}\left(\frac{F_*\Omega_R^{[i]}}{B\Omega_R^{[i]}}\right)$$

is injective for all $0 \leq i \leq k$, $0 \leq i+j \leq d$, *and all maximal ideals* $\mathfrak{m} \in \mathrm{MaxSpec}(R)$.

For $i = 0$, $\Omega_R^{[0]} = R$ and $B\Omega_R^{[0]} = 0$; it follows that $(0, F)$-injectivity is equivalent to R being F-injective and the condition strengthens with k in the sense that (k, F)-injectivity implies $(k-1, F)$-injectivity. In particular, if R is quasi-Gorenstein then R (k, F)-injective implies R is F-split.

Remark 2.9. Just as F-injectivity can be viewed as an analogue of Du Bois singularities in positive characteristic (see [Sch09, BST17]), (k, F)-injective singularities are a characteristic p analogue of k-Du Bois singularities, see [KW24, Theorem A].

In practice, as we saw in Remark 2.4, the reflexivity condition on Ω_R^i is a strong hypothesis. A weaker version yields the notion of *pre-*(k, F)-injective singularities:

Definition 2.10. *Let* R *be a normal* d*-dimensional domain of finite type over a perfect field* k *of characteristic* $p > 0$. *We say* R *is* pre-(k, F)-injective *if*

(i) $C : Z\Omega_R^{[i]} \to \Omega_R^{[i]}$ *is surjective for* $i \leq k$
(ii)

$$C^{-1} : \mathrm{H}^j_{\mathfrak{m}}\left(\Omega_R^{[i]}\right) \to \mathrm{H}^j_{\mathfrak{m}}\left(\frac{F_*\Omega_R^{[i]}}{B\Omega_R^{[i]}}\right)$$

is injective for all $0 \leq i \leq k$, $0 \leq i+j \leq d$, *and all maximal ideals* $\mathfrak{m} \in \mathrm{MaxSpec}(R)$.

By Lemma 2.3, (k, F)-injective implies pre-(k, F)-injective; the benefit of this definition is that it works when Ω_R^i are not necessarily reflexive. Note that for $k > 0$ this notion is a priori different from the definition of pre-(k, F)-injectivity in [KW24, Def. 3.18], but the two definitions coincide when the associated eh-sheaves equal their reflexivizations, i.e. $\Omega^i_{R,eh} \simeq \Omega_R^{[i]}$ for $i \leq k$ ([KW24, Lemma 3.11, Prop. 3.20]). We do not know whether one definition implies the other in general.

We next define (k, F)-rationality. We retain the normality assumptions on $X = \mathrm{Spec}(R)$, but as in Definition 2.10 we ask for the reflexive Cartier operator $C : Z\Omega_R^{[i]} \to \Omega_R^{[i]}$ to be surjective. By Lemma 2.6 we can then view C_n^{-1} as a map $\Omega_R^{[i]} \hookrightarrow F_*^n\Omega_R^{[i]}/B_n\Omega_R^{[i]}$.

Definition 2.11 ([KW26], §3)**.** *Let* R *be a normal* d*-dimensional ring of finite type over a perfect field* k *of characteristic* p *and* $k \geq 0$ *is fixed. We say* R *is* (k, F)-rational *if*

(i) $C : Z\Omega_R^{[i]} \to \Omega_R^{[i]}$ *is surjective for* $i \leq k$
(ii) $\mathrm{codim}_R(\mathrm{Sing}(R)) \geq 2k+2$
(iii) *For every closed* $\mathfrak{m} \in X = \mathrm{Spec}(R)$ *and* $D \geq 0$ *Cartier divisor on* X, *there exists an* m *so that the composite*

$$\mathrm{H}^j_{\mathfrak{m}}\left(\Omega_R^{[i]}\right) \xrightarrow{C_n^{-1}} \mathrm{H}^j_{\mathfrak{m}}\left(\frac{F_*^n\Omega_R^{[i]}}{B_n\Omega_R^{[i]}}\right) \longrightarrow \mathrm{H}^j_{\mathfrak{m}}\left(\frac{F_*^n(\Omega_R^{[i]}(D))}{B_n\Omega_R^{[i]}}\right)$$

is injective for all $n \geq m$, $i \leq k$ *and* $i + j \leq d$.

If we drop the codimension condition (ii), R is called *pre-*(k,F)-rational. We only consider isolated singularities in this paper, whence (ii) translates to $k \leq \lfloor d/2-1 \rfloor$. We also recall two equivalent definitions of (k,F)-rationality below:

Proposition 2.12 ([KW26], Corollary 3.2, Remark 3.5)**.** *Suppose R is a normal d-dimensional ring with an isolated singularity at $\mathfrak{m} \in X = \operatorname{Spec}(R)$ and $k \leq \lfloor d/2-1 \rfloor$ is fixed. Then R is (k,F)-rational if and only if:*

(i) *X is F-rational.*

(ii) *There exists $m > 0$ such that $\mathrm{H}^j_{\mathfrak{m}}(Z_n\Omega^{[i]}_X) = 0$ for all $n \geq m$, $0 < i \leq k$ and $i + j \leq d$. Equivalently, the following composite map is injective for the same region of n, i, j:*

$$\mathrm{H}^j_{\mathfrak{m}}(\Omega^{[i]}_R) \xrightarrow{C_n^{-1}} \mathrm{H}^j_{\mathfrak{m}}\left(\frac{F^n_*\Omega^{[i]}_X}{B_n\Omega^{[i]}_R}\right) \xrightarrow{\delta} \mathrm{H}^{j+1}_{\mathfrak{m}}(B_n\Omega^{[i]}_R)$$

Here δ is the boundary map from exact sequence $B_n\Omega^{[i]}_R \to F^n_*\Omega^{[i]}_R \to F^n_*\Omega^{[i]}_R/B_n\Omega^{[i]}_R$.

As with (k,F)-injectivity, we have that a (k,F)-rational ring R is also $(k-1,F)$-rational. Since (k,F)-rationality forces C_n^{-1} to be injective for $n \gg 0$, and C_n^{-1} factors through $\Omega^{[i]}_R \to F_*\Omega^{[i]}_R/B\Omega^{[i]}_R$, we get that R is pre-(k,F)-injective as well. We also note that (k,F)-rational singularities are Cohen–Macaulay since they are by definition F-rational.

Example 2.13. The following cases illustrate the difference between higher F-singularities and their pre-variants.

(i) The threefold node $R = k[x,y,z,w]/(xw-yz)$ is pre-(k,F)-injective for all k as F-liftability implies the Cartier map is split surjective, and one can check it is $(1,F)$-injective ([KW24, Lemma 4.1 and Rem. 1.4]). Similarly A_1 surface singularity $R = k[x,y,z]/(xy-z^2)$ is pre-(k,F)-injective and F-injective, but not $(1,F)$-injective.

(ii) Assume $p > 2$. The Veronese cone $R = C(\mathbb{P}^3, \mathcal{O}(2)) \simeq \mathbb{A}^4/\mu_2$ is pre-(k,F)-rational for all k and (k,F)-rational only for $k \leq 1$ ([KW26, Ex. 3.9]).

3. On various notions of k-ordinarity

3.1. **De Rham cohomology, crystalline cohomology and Mazur–Ogus varieties.** Let k be a perfect field of characteristic p and $W = W(k), W_n = W/p^nW$ its ring of Witt vectors and n-truncated Witt vectors respectively.

Recall that for a smooth proper variety X/k, its de Rham cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}(X/k)$ is the hypercohomology of its de Rham complex $\Omega^\bullet_{X/k}$. There are two spectral sequences converging to $\mathrm{H}^\bullet_{\mathrm{dR}}(X/k)$: the first is the Hodge-to-de Rham spectral sequence

$$E_1^{ij} = \mathrm{H}^j(X, \Omega^i_X) \Longrightarrow \mathrm{H}^{i+j}_{\mathrm{dR}}(X/k)$$

and the second is the conjugate spectral sequence

$$E_2^{ij} = \mathrm{H}^i(X, \mathcal{H}^j(F_*\Omega^\bullet_X)) \Longrightarrow \mathbb{H}^{i+j}(X, F_*\Omega^\bullet_X) = \phi_* H^{i+j}_{\mathrm{dR}}(X/k)$$

where ϕ is the Frobenius on $\operatorname{Spec}(k)$ and the k-vector space structure on $\phi_* H^{i+j}_{\mathrm{dR}}(X/k)$ is $\lambda \cdot v = \lambda^p v$ for $\lambda \in k, v \in \mathrm{H}^n_{\mathrm{dR}}(X/k)$. Via the Cartier isomorphism (i.e. the global analogue of Lemma 2.1), $\mathcal{H}^j(F_*\Omega^\bullet_X) \simeq \Omega^j_X$ and so the Hodge-to-de Rham spectral sequence degenerates at E_1 if and only if the conjugate spectral sequence degenerates at E_2.

The two spectral sequences respectively induce two abutment filtrations on $\mathrm{H}^n_{\mathrm{dR}}(X/k)$. Writing $n = i + j$ we have

$$F^i = \operatorname{im}\left(\mathbb{H}^n(X, \Omega_X^{\bullet \geq i}) \to \mathrm{H}^n_{\mathrm{dR}}(X/k)\right) \tag{10}$$

$$G_i = \operatorname{im}\left(\mathbb{H}^n(X, \tau_{\leq i}(F_*\Omega_X^\bullet)) \to \mathrm{H}^n_{\mathrm{dR}}(X/k)\right) \tag{11}$$

where $\tau_{\leq i}(F_*\Omega_X^\bullet) = [F_*\mathcal{O}_X \to \cdots \to F_*\Omega_X^{i-1} \to Z\Omega_X^i \to 0]$ is the canonical truncation; $F^\bullet$ is commonly known as the *Hodge* filtration while $G_\bullet$ is known as the *conjugate* filtration. We note that the target in (11) is $\phi_* \mathrm{H}^n_{\mathrm{dR}}(X/k)$, which by transport of structure we view as the k-vector space $\mathrm{H}^n_{\mathrm{dR}}(X/k)$.

To a smooth proper X we can also attach its crystalline cohomology groups $\mathrm{H}^\bullet_{\mathrm{crys}}(X/W)$, which are graded W-modules functorially dependent on X. They relate to its de Rham cohomology via exact sequence

$$0 \to \mathrm{H}^n_{\mathrm{crys}}(X/W) \otimes_W k \to \mathrm{H}^n_{\mathrm{dR}}(X/k) \to \mathrm{H}^{n+1}_{\mathrm{crys}}(X/W)[p] \to 0 \tag{12}$$

where $\mathrm{H}^{n+1}_{\mathrm{crys}}(X/W)[p] = \operatorname{Tor}_1^W(\mathrm{H}^{n+1}_{\mathrm{crys}}(X/W), k)$ denotes p-torsion.

By functoriality, the absolute Frobenius on X induces the "crystalline Frobenius"

$$\phi : \mathrm{H}^n_{\mathrm{crys}}(X/W) \to \mathrm{H}^n_{\mathrm{crys}}(X/W)$$

which is semilinear with respect to the Frobenius σ on W and an isogeny, i.e. $\phi \otimes \mathbb{Q}$ is bijective. Thus $\mathrm{H}^n_{\mathrm{crys}}(X/W)_{\mathrm{tf}} := \mathrm{H}^n_{\mathrm{crys}}(X/W)/(\text{torsion})$ has a natural F-crystal structure and we may associate two invariants to it, the Hodge and Newton polygons, which we now explain.

- Let $s_n = \sum h^{i,n-i}$ where $h^{i,n-i} = \dim_k \mathrm{H}^{n-i}(X, \Omega_X^i)$. The *Hodge polygon* associated to $\mathrm{H}^n_{\mathrm{crys}}(X/W)_{\mathrm{tf}}$ is the graph of the piecewise-linear function $h : [0, s_n] \to \mathbb{R}$ with $h(0) = 0$ and with slope i of multiplicity $h^{i,n-i}$. Equivalently it is a polygonal line starting at $(0, 0)$ with vertices
$$\left(\sum_{j=0}^{i} h^{j,n-j}, \sum_{j=0}^{i} j h^{j,n-j}\right)$$
- By the Dieudonné–Manin classification, $\mathrm{H}^n_{\mathrm{crys}}(X/W)_{\mathrm{tf}} \otimes_W W(\overline{k})$ is isogenous to a direct sum of F-crystals of the form
$$\frac{W(\overline{k})[T]}{T^b - p^a} \tag{13}$$
where ϕ acts as multiplication by T (in the sense that $\phi(f(T)) = T\sigma(f)(T)$ with σ acting on the coefficients of f) and $\lambda = a/b \geq 0$, $(a, b) = 1$. The λ that appear in this decomposition are called the slopes of $\mathrm{H}^n_{\mathrm{crys}}(X/W)_{\mathrm{tf}}$, and their multiplicities $\mathrm{mult}(\lambda)$ are b times the number of factors of form (13) with $a/b = \lambda$ that appear in the decomposition.

  Write $r = \operatorname{rk} \mathrm{H}^n_{\mathrm{crys}}(X/W)_{\mathrm{tf}}$. The *Newton polygon* is the graph of the piecewise-linear function $h : [0, r] \to \mathbb{R}$ with $h(0) = 0$ and with slope λ_i of multiplicity $\mathrm{mult}(\lambda_i)$, where $0 \leq \lambda_1 \leq \cdots \leq \lambda_r$. When $k = \mathbb{F}_q$ with $q = p^e$, the slopes of the Newton polygon are $v_p(\alpha)/e$ (counted with multiplicity) where α runs through the eigenvalues of the endomorphism ϕ^e of $\mathrm{H}^n_{\mathrm{crys}}(X/W)_{\mathrm{tf}}[1/p]$, normalized so that $v_p(p) = 1$.

By a theorem of Mazur–Ogus ([BO78, Thm. 8.39]), under the assumption of torsionfree crystalline cohomology and E_1-degeneration of the Hodge-to-de Rham spectral sequence of X, the Newton polygon of $\mathrm{H}^n_{\mathrm{crys}}(X/W)$ lies on or above its Hodge polygon and they have the same endpoint with horizontal value $r = \operatorname{rk} \mathrm{H}^n_{\mathrm{crys}}(X/W) = \dim_k \mathrm{H}^n_{\mathrm{dR}}(X/k)$. In fact, the conditions of the theorem of Mazur–Ogus are ubiquitous so we isolate them into a separate definition:

Definition 3.1. *A smooth proper variety is called* Mazur–Ogus *if*

(i) $\mathrm{H}^i_{crys}(X/W)$ *is torsion free for all* i.
(ii) *The Hodge-to-de Rham spectral sequence degenerates at* E_1.

A further motivation for this definition is that the Hodge numbers $h^{i,j}(X) := \dim H^j(X, \Omega^i_X)$ share some properties similar to the Hodge numbers of characteristic 0 varieties. To be precise, there is a canonical isomorphism $\mathrm{H}^i_{\mathrm{crys}}(X/W) \otimes_W k \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(X)$ by (i), and so by (ii) the crystalline Betti numbers are $b^{\mathrm{crys}}_n(X) := \operatorname{rk}_W \mathrm{H}^n_{\mathrm{crys}}(X/W) = \sum_{i+j=n} h^{ij}$.

Example 3.2. Smooth projective curves (of any genus), abelian varieties, K3 surfaces and smooth complete intersections in $\mathbb{P}^n$ are Mazur–Ogus varieties. One can construct several more classes of examples by noting that products of Mazur–Ogus varieties and blowups of Mazur–Ogus varieties along smooth Mazur–Ogus centers are also Mazur–Ogus (cf. [AZ17, Corollary 2.9]).

Remark 3.3. There are examples of liftable and even F-liftable varieties that are not Mazur–Ogus. A singular Enriques surface in characteristic 2 admits a smooth lift to $W(k)$ but has nontrivial torsion in $\mathrm{H}^2_{\mathrm{crys}}$ ([Ill79, Corollaire 7.4.3 (a)]). The Igusa surface in characteristic 2 constructed in [Jos21, Remark 2.12] has torsion in $\mathrm{H}^2_{\mathrm{crys}}$ but it is a quotient of an ordinary abelian surface by a free action, hence F-liftable by [MS87b, Theorem 2 and Appendix, Proposition 1].

Mazur–Ogus varieties do *not* in general satisfy Hodge symmetry ($h^{ij} = h^{ji}$). As an example, using Petrov's construction ([Pet21, §4]), we let X be the special fiber of a proper smooth formal scheme $\mathfrak{X}/\mathrm{Spf}(\mathbb{Z}_p)$, and $\mathfrak{X}_\eta$ is the rigid analytic generic fiber. Then $h^{ij}(X) = h^{ij}(\mathfrak{X}_\eta)$ via torsionfreeness of the integral Hodge cohomology groups and base-change, and so by the crystalline-de Rham comparison we have

$$b^{\mathrm{crys}}_n(X) = b^{\text{ét}}_n(\mathfrak{X}_\eta) = \sum_{i+j=n} h^{ij}(\mathfrak{X}_\eta) = \sum_{i+j=n} h^{ij}(X)$$

from which we get that X is Mazur–Ogus. However there are $\mathfrak{X}$ such that $h^{ij}(\mathfrak{X}_\eta) \neq h^{ji}(\mathfrak{X}_\eta)$, and so X cannot satisfy Hodge symmetry. We can remedy this as follows: if X is Mazur–Ogus and admits a smooth proper algebraic lift to characteristic zero, then it also satisfies Hodge symmetry. In this case, $\mathfrak{X}_\eta$ is a smooth algebraic variety over characteristic zero, which satisfies Hodge symmetry, so by $h^{ij}(X) = h^{ij}(\mathfrak{X}_\eta)$ we get that X also satisfies Hodge symmetry.

3.2. **Comparisons between k-ordinary varieties.**

Definition 3.4. *Let X be a smooth projective variety of dimension d over a perfect field of characteristic p and set $V_n = \mathrm{H}^n_{\mathrm{crys}}(X/W)$. For each k, we say X is:*

(i) Bloch–Kato k-ordinary *if* $\mathrm{H}^j(X, B\Omega^i_X) = 0$ *for $i \leq k$ and all $j \geq 0$. X is* ordinary *if it is Bloch–Kato k-ordinary for all k.*

(ii) skew k-ordinary in degree n *if* $\mathrm{H}^j(X, B\Omega^i_X) = 0$ *for* $i \leq k$ *and* $i + j \leq n$. X *is* skew k-ordinary *if it is skew* k*-ordinary in degree* d.

(iii) Newton–Hodge k-ordinary in degree n *if the slope* $[0, k)$ *part of the Newton and Hodge polygon of* V_n *coincide. If the Newton and Hodge polygons of* V_n *coincide, we call* X Newton–Hodge *ordinary in degree* n.

When X is an abelian variety, ordinarity in the sense of (i) above is just classical ordinarity, i.e. X has maximal p-rank $X[p](\overline{k}) \simeq (\mathbb{Z}/p\mathbb{Z})^{\dim(X)}$. In fact an abelian variety is ordinary in the classical sense if and only if it is Bloch–Kato 1-ordinary. When X is a smooth curve of genus g, it is ordinary exactly when its Jacobian $\mathrm{Jac}(X)$ has p-rank g. Thus, much like abelian varieties, curves are ordinary if and only if they are Bloch–Kato 1-ordinary. Let us also mention here that ordinary varieties satisfy (i)-(iii) for all k and degrees n when they have torsionfree crystalline cohomology, so there is no ambiguity of what 'ordinary' means in this case.

Remark 3.5 (Comparing different notions of ordinarity)**.** By definition, Bloch–Kato k-ordinarity implies skew k-ordinarity in all degrees n. For X Mazur–Ogus, Newton–Hodge k-ordinarity in all degrees n is equivalent to Bloch–Kato k-ordinarity ([BT25, Prop. 8.9]). We also have the following observations as k varies:

(i) When $k = 1$, one can check that X is Bloch–Kato 1-ordinary if and only if it is skew 1-ordinary, if and only if $F : \mathrm{H}^i(X, \mathcal{O}_X) \to \mathrm{H}^i(X, F_*\mathcal{O}_X)$ is an isomorphism for all i. For $i = d$, this condition is known as X being *weakly* ordinary[1]. When X is Calabi–Yau, weakly ordinary coincides with the notions of Bloch–Kato and skew 1-ordinary, is equivalent to X being F-split and agrees with Achinger–Zdanowicz's definition of 1-ordinarity ([AZ21, Def. 2.2.2]).

(ii) For $k \geq 2$, Bloch–Kato k-ordinarity need not be the same as skew k-ordinarity (see Proposition 3.8 for when these notions coincide). When the variety is Mazur–Ogus and has trivial canonical bundle, Achinger–Zdanowicz's version of k-ordinarity is the same as Newton–Hodge k-ordinarity in degree d ([AZ21, Prop. 2.3.1.(3)]).

We now investigate more closely the relationship between skew k-ordinarity and Newton–Hodge or Bloch–Kato k-ordinarity, subject to special assumptions on the variety X. We first have the following lemma:

Lemma 3.6. *Suppose* X *satisfies* E_1*-degeneration of its Hodge-to-de Rham spectral sequence and* $\mathrm{H}^j(X, B\Omega^i_X) = 0$ *for fixed* i, j *with* $i + j = n$. *Then we have an identification of* $\mathbb{F}_p$*-vector spaces*

$$G_{i-1}\,\mathrm{H}^n_{dR}(X/k) \oplus F^i\,\mathrm{H}^n_{dR}(X/k) = \mathrm{H}^n_{dR}(X/k)$$

where $G_\bullet, F^\bullet$ *denote the conjugate and Hodge filtration respectively.*

Proof. Consider exact triangle

$$\tau_{\leq i-1}(F_*\Omega^\bullet_X) \oplus F_*\Omega^{\geq i}_X \xrightarrow{f} F_*\Omega^\bullet_X \to \mathrm{Cone}(f) \xrightarrow{[1]} \tag{14}$$

and note the leftmost complex is $Z\Omega^{i-1}_X$ in degree $i - 1$ and $F_*\Omega^k_X$ in all other degrees k, hence $\mathrm{Cone}(f)$ is quasi-isomorphic to $F_*\Omega^{i-1}_X/Z\Omega^{i-1}_X[1 - i] \simeq B\Omega^i_X[1 - i]$. We have another

[1]The notion of weak ordinarity is somewhat inconsistent in the literature; what we call "weakly ordinary" here is consistent with the statement of the weak ordinarity conjecture (cf. [BST17, Conj. 1.1]).

natural exact triangle

$$\tau_{\leq i-1}(F_*\Omega_X^\bullet) \to F_*\Omega_X^\bullet \to \tau_{\geq i}(F_*\Omega_X^\bullet) \xrightarrow{[1]}$$

which upon taking hypercohomology yields

$$\cdots \to \mathbb{H}^n(X, \tau_{\leq i-1}(F_*\Omega_X^\bullet)) \to \phi_* \operatorname{H}^n_{\mathrm{dR}}(X/k) \to \mathbb{H}^n(X, \tau_{\geq i}(F_*\Omega_X^\bullet)) \to \cdots \tag{15}$$

where we have identified $\phi_* \operatorname{H}^n_{\mathrm{dR}}(X/k) = \mathbb{H}^n(X, F_*\Omega_X^\bullet)$. Now since X has its Hodge-to-de Rham spectral sequence degenerating at E_1, whence $\dim(\phi_* \operatorname{H}^n_{\mathrm{dR}}(X/k)) = \sum_{k\leq n} h^{k,n-k}(X)$. By the conjugate spectral sequence we also have

$$\dim \mathbb{H}^n(X, \tau_{\leq i-1}(F_*\Omega_X^\bullet)) \leq \sum_{k\leq i-1} \dim \operatorname{H}^{n-k}(X, \mathcal{H}^k(F_*\Omega_X^\bullet)) = \sum_{k\leq i-1} h^{k,n-k}(X)$$

$$\dim \mathbb{H}^n(X, \tau_{\geq i}(F_*\Omega_X^\bullet)) \leq \sum_{k\geq i} h^{k,n-k}(X)$$

thus exactness of (15) in the middle and a dimension count yields $\dim \mathbb{H}^n(X, \tau_{\leq i-1}(F_*\Omega_X^\bullet)) = \sum_{k\leq i-1} h^{k,n-k}$ and that $\mathbb{H}^n(X, \tau_{\leq i-1}(F_*\Omega_X^\bullet))$ injects into $\operatorname{H}^n_{\mathrm{dR}}(X/k)$, hence is identified with $G_{i-1} \operatorname{H}^n_{\mathrm{dR}}(X/k)$. A similar argument applied to complex $F_*\Omega_X^{\bullet \geq i}$ yields $\mathbb{H}^n(X, F_*\Omega_X^{\bullet\geq i}) = \phi_* F^i \operatorname{H}^n_{\mathrm{dR}}(X/k)$. Suppressing the ϕ_* notation and taking hypercohomology in (14) yields

$$\operatorname{H}^{n-1}(X, B\Omega_X^i[1-i]) \to G_{i-1} \operatorname{H}^n_{\mathrm{dR}}(X/k) \oplus F^i \operatorname{H}^n_{\mathrm{dR}}(X/k) \to \operatorname{H}^n_{\mathrm{dR}}(X/k) \to \cdots$$

where the leftmost vector space is identified with $\operatorname{H}^j(X, B\Omega_X^i) = 0$ by assumption. Hence $G_{i-1} \operatorname{H}^n_{\mathrm{dR}}(X/k) \oplus F^i \operatorname{H}^n_{\mathrm{dR}}(X/k)$ injects into $\operatorname{H}^n_{\mathrm{dR}}(X/k)$ and has dimension $\sum_{k\leq n} h^{k,n-k}(X)$, i.e. $G_{i-1} \operatorname{H}^n_{\mathrm{dR}}(X/k) \oplus F^i \operatorname{H}^n_{\mathrm{dR}}(X/k) = \operatorname{H}^n_{\mathrm{dR}}(X/k)$. □

Proposition 3.7. *If X is Mazur–Ogus, then X is skew k-ordinary if and only if the slope $[0, k)$ segments of the Hodge and Newton polygon of $\operatorname{H}^n_{crys}(X/W)$ coincide for all $n \leq d$.*

Proof. By the exact triangle (14) in the proof of Lemma 3.6 and the Mazur–Ogus hypothesis, we have a long exact sequence

$$G_{i-1} \operatorname{H}^{n-1}_{\mathrm{dR}} \oplus F^i \operatorname{H}^{n-1}_{\mathrm{dR}} \xrightarrow{f^i_{n-1}} \operatorname{H}^{n-1}_{\mathrm{dR}} \to \operatorname{H}^{n-i}(X, B\Omega_X^i) \to G_{i-1} \operatorname{H}^n_{\mathrm{dR}} \oplus F^i \operatorname{H}^n_{\mathrm{dR}} \xrightarrow{f^i_n} \operatorname{H}^n_{\mathrm{dR}} \tag{16}$$

If X is skew k-ordinary then $\operatorname{H}^{n-i}(X, B\Omega_X^i) = 0$ for $i \leq k, n \leq d$ so f^i_n is injective and f^i_{n-1} is surjective. Since by E_1-degeneration we have $\dim(G_{i-1} \operatorname{H}^n_{\mathrm{dR}} \oplus F^i \operatorname{H}^n_{\mathrm{dR}}) = \dim \operatorname{H}^n_{\mathrm{dR}}$ for $n \leq d$ we obtain that f^i_n are isomorphisms for $i \leq k, n \leq d$. By the proof in [DI06, Proposition 1.3.2 (i), (ii)] and the identification of the Hodge and conjugate filtrations on F-crystals in *loc. cit* with the usual ones on de Rham cohomology ([BO78, Theorem 8.26]), we see that f^i_n are isomorphisms if and only if the first k segments of the Newton and Hodge polygons of $\operatorname{H}^n_{\mathrm{crys}}(X/W)$ coincide (cf. [AZ21, Prop. 2.3.1 (2)]).

Conversely, if the slopes of the Newton and Hodge polygons coincide in $[0, k)$ for $\operatorname{H}^n_{\mathrm{crys}}(X/W)$, $n \leq d$, then by E_1-degeneration and [DI06] as before we see that f^i_n are isomorphisms for $i \leq k, n \leq d$, hence (16) yields $\operatorname{H}^j(X, B\Omega_X^i) = 0$ for $i \leq k$ and $i + j \leq d$. □

More generally, Proposition 3.7 shows that a Mazur–Ogus variety X is skew k-ordinary in degree n if and only if it is Newton–Hodge k-ordinary in *all* degrees $\leq n$.

As another application of Lemma 3.6, we can detect when skew k-ordinarity implies Bloch–Kato k-ordinarity as follows:

Proposition 3.8. *Suppose X is Mazur–Ogus and also satisfies Hodge symmetry. Then X is skew k-ordinary if and only if it is Bloch–Kato k-ordinary.*

Proof. Since the 'if' direction is true by definition, we need only focus on X being skew k-ordinary. By Proposition 3.7, the slope $[0,k)$ segments of the Hodge and Newton polygon of $\mathrm{H}^n_{\mathrm{crys}}(X/W)$ coincide for all degrees n up to d. Now via the crystalline Hard Lefschetz we have an isomorphism

$$c_a : \mathrm{H}^{d-a}_{\mathrm{crys}}(X/W) \otimes_W K \xrightarrow{\sim} \mathrm{H}^{d+a}_{\mathrm{crys}}(X/W) \otimes_W K \tag{17}$$

given by the a-fold cup product with the crystalline Chern class $c_1(L)$ of an ample line bundle L on X, where $K = \mathrm{Frac}W$. See [KM74, Corollary 1.(2)] for the formulation of the isomorphism when k is finite; this extends to k perfect. By virtue of $\phi(c_1(L)) = c_1(F^*L) = c_1(L^p) = pc_1(L)$, compatibility of the isomorphism with the crystalline Frobenius ϕ yields $V_{d-a}(-a) \simeq V_{d+a}$, where $V_n = \mathrm{H}^n_{\mathrm{crys}}(X/W) \otimes_W K$. Thus for a slope λ_i of F-crystal V_{d-a}, there is a slope $\lambda_i + a$ of V_{d+a} with the same multiplicity.

Choose $n > d$ so that the isomorphism in (17) gives $V_n \simeq V_{2d-n}(d-n)$. Denoting by λ_i the slopes of V_n, we get that the slopes of V_{2d-n} are $\lambda_i + d - n$, of the same multiplicity as λ_i. For Hodge cohomology, recall that slope i of the Hodge polygon of $\mathrm{H}^n_{\mathrm{crys}}(X/W)$ has multiplicity $h^{i,n-i}$, and slope $i+d-n$ of the Hodge polygon of $\mathrm{H}^{2d-n}_{\mathrm{crys}}(X/W)$ has multiplicity

$$h^{i+d-n,d-i} = h^{n-i,i} = h^{i,n-i}$$

where the first equality follows from Serre duality, and the second equality follows from Hodge symmetry. Thus slope $i+d-n$ has the same multiplicity as slope i in $\mathrm{H}^n_{\mathrm{crys}}(X/W)$.

Since $2d-n < d$, we get coincidence of slopes in $[0,k)$ for the Hodge and Newton polygon of $\mathrm{H}^{2d-n}_{\mathrm{crys}}(X/W)$ from skew k-ordinarity. Thus, for the slopes of the Newton and Hodge polygons of $\mathrm{H}^n_{\mathrm{crys}}(X/W)$, coincidence in the interval $[0,k)$ follows from coincidence of the slopes of $\mathrm{H}^{2d-n}_{\mathrm{crys}}(X/W)$ in the interval $[0, k+d-n)$ which is contained in $[0,k)$.

We conclude that X is Newton–Hodge k-ordinary in all degrees $n > d$ as well, which is equivalent to X being Bloch–Kato k-ordinary (cf. Remark 3.5). □

For special classes of varieties where the only 'interesting' cohomology is concentrated in the middle degree d, the various notions of k-ordinarity that we consider are equivalent. This includes several classes of varieties such as curves, rational surfaces, K3 surfaces, complete intersection Calabi–Yau threefolds and cubic fourfolds, and Fano varieties of K3 types among others. To make this statement precise:

Corollary 3.8.1. *Assume X is Newton–Hodge ordinary in all degrees $n \neq d$. Then the following are equivalent:*

(i) *X is skew k-ordinary*
(ii) *X is Bloch–Kato k-ordinary*
(iii) *X is Newton–Hodge k-ordinary in degree d*

Proof. We first explain why X is Mazur–Ogus. Recall $b^{\mathrm{crys}}_n(X) = \mathrm{rk}_W \mathrm{H}^n_{\mathrm{crys}}(X/W)$ and set $s_n(X) = \sum_{i+j=n} h^{ij}(X)$, then by Newton–Hodge ordinarity in degrees $n \neq d$ we have $b^{\mathrm{crys}}_n(X) = s_n(X)$ for $n \neq d$ as the horizontal endpoints of the Newton and Hodge polygons of $\mathrm{H}^n_{\mathrm{crys}}(X/W)$. Recall the exact sequence from (12):

$$0 \to \mathrm{H}^n_{\mathrm{crys}}(X/W) \otimes_W k \to \mathrm{H}^n_{\mathrm{dR}}(X/k) \to \mathrm{H}^{n+1}_{\mathrm{crys}}(X/W)[p] \to 0$$

Since

$$\dim \mathrm{H}^n_{\mathrm{dR}}(X/k) = \dim_k(\mathrm{H}^n_{\mathrm{crys}}(X/W) \otimes_W k) + \dim_k(\mathrm{H}^{n+1}_{\mathrm{crys}}(X/W)[p]) \leq s_n(X),$$

$$\dim_k(\mathrm{H}^n_{\mathrm{crys}}(X/W)\otimes_W k) \geq b^n_{\mathrm{crys}}(X) = s_n(X)$$

for $n \neq d$, we get that $\mathrm{H}^n_{\mathrm{crys}}(X/W), \mathrm{H}^{n+1}_{\mathrm{crys}}(X/W)$ are torsionfree and $\dim \mathrm{H}^n_{\mathrm{dR}}(X/k) = s_n(X)$ for $n \neq d$; in particular for $n = d-1$ we get $\mathrm{H}^d_{\mathrm{crys}}(X/W)$ torsionfree, so overall X has torsionfree crystalline cohomology.

To check E_1-degeneration, set $e_{r,n} = \Sigma_{i+j=n}\dim(E_r^{ij})$ and note by a spectral sequence computation that $\sum_n (-1)^n e_{r,n}$ stays constant for $r = 1, 2, \cdots$. For $r = 1$ we have $e_{1,n} = \sum_{i+j=n} h^{ij}(X) = s_n(X)$. For $r = \infty$ we have

$$e_{\infty,n} = \Sigma_{i+j=n}\dim(E_\infty^{ij}) = \Sigma_{i+j=n}\dim\left(F^i\,\mathrm{H}^n_{\mathrm{dR}}(X/k)/F^{i+1}\,\mathrm{H}^n_{\mathrm{dR}}(X/k)\right) = \dim \mathrm{H}^n_{\mathrm{dR}}(X/k)$$

by exhaustiveness of the Hodge filtration $F^\bullet$. Thus

$$\sum_n (-1)^n s_n(X) = \sum_n (-1)^n \dim \mathrm{H}^n_{\mathrm{dR}}(X/k)$$

and $s_n(X) = \dim \mathrm{H}^n_{\mathrm{dR}}(X/k)$ for $n \neq d$ forces equality for $n = d$ as well, so X is Mazur–Ogus.

Since X is Mazur–Ogus, Proposition 3.7 gives the equivalence of (i) and (iii). Bloch–Kato k-ordinarity is the same as Newton–Hodge k-ordinarity in all degrees, so (ii) is automatically equivalent to (iii). □

The case of cubic fourfolds is particularly relevant for this paper (see Appendix A), so we record it separately here.

Lemma 3.9. *For a cubic fourfold X the following are equivalent:*

(i) *X is ordinary*
(ii) *X is skew 2-ordinary*
(iii) $\mathrm{H}^2(X, B\Omega^2_X) = 0$
(iv) $\mathrm{H}^3(X, B\Omega^2_X) = 0$

Proof. Let us first determine for which i, j we have $H^j(X, B\Omega^i_X) = 0$ unconditionally. Since X is Mazur–Ogus, the Hodge diamond of X coincides with that of the generic fiber $\mathcal{X}_K$ of a smooth lift of X over $W(k)$, where $K = \mathrm{Frac}(W(k))$. Hence $h^{ij}(X) = 1$ for $(i,j) = (0,0), (1,1), (3,3), (4,4), (1,3), (3,1)$, $h^{22}(X) = 21$ and otherwise $h^{ij}(X) = 0$. Now via exact sequence $0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B\Omega^1_X \to 0$ we obtain

$$\cdots \mathrm{H}^j(X, \mathcal{O}_X) \to \mathrm{H}^j(X, F_*\mathcal{O}_X) \to \mathrm{H}^j(X, B\Omega^1_X) \to \mathrm{H}^{j+1}(X, \mathcal{O}_X) \to \cdots$$

and so by looking at the Hodge diamond of X we get $\mathrm{H}^j(X, B\Omega^1_X) = 0$ for $j \in [0,4]$; by Serre duality we also obtain $\mathrm{H}^j(X, B\Omega^4_X) = 0$ for $j \in [0,4]$.

We next have $\mathrm{H}^0(X, B\Omega^2_X) \hookrightarrow \mathrm{H}^0(X, \Omega^2_X) = 0$ and $\mathrm{H}^0(X, B\Omega^3_X) \hookrightarrow \mathrm{H}^0(X, \Omega^3_X) = 0$ so by Serre duality (see Proposition 2.7) we also have $\mathrm{H}^4(X, B\Omega^2_X) = \mathrm{H}^4(X, B\Omega^3_X) = 0$. To check the vanishing of $\mathrm{H}^1(X, B\Omega^2_X)$, use the two exact sequences:

$$0 \to B\Omega^2_X \to Z\Omega^2_X \xrightarrow{C} \Omega^2_X \to 0, \;\; 0 \to Z\Omega^2_X \to F_*\Omega^2_X \to B\Omega^3_X \to 0$$

From the first sequence we get $\mathrm{H}^1(X, B\Omega^2_X) \simeq \mathrm{H}^1(X, Z\Omega^2_X)$ and from the second sequence we get $0 = \mathrm{H}^0(X, B\Omega^3_X) \to \mathrm{H}^1(X, Z\Omega^2_X) \to \mathrm{H}^1(X, F_*\Omega^2_X) = 0$ exact i.e. $\mathrm{H}^1(X, Z\Omega^2_X) = 0$, hence also $\mathrm{H}^3(X, B\Omega^3_X) = 0$ by Serre duality.

In summary, $\mathrm{H}^j(X, B\Omega^i_X) = 0$ except possibly for $(i,j) = (2,2), (2,3), (3,2), (3,1)$; the latter two pairs are Serre dual to the first two. Implications (i) ⇒ (ii) ⇒ (iii) are obvious, so we check (iii) ⇔ (iv). Similar to the above, we use two exact sequences

$$0 \to B\Omega^3_X \to Z\Omega^3_X \xrightarrow{C} \Omega^3_X \to 0, \;\; 0 \to Z\Omega^3_X \to F_*\Omega^3_X \to B\Omega^4_X \to 0$$

and observe that, since $\mathrm{H}^j(X, B\Omega^4_X) = 0$, we get $\mathrm{H}^1(X, Z\Omega^3_X) \simeq \mathrm{H}^1(X, F_*\Omega^3_X) \simeq k$ by the second sequence. Then the first one yields $\mathrm{H}^1(X, B\Omega^3_X) = \ker(C)$, $\mathrm{H}^2(X, B\Omega^3_X) = \operatorname{coker}(C)$, where $C : \mathrm{H}^1(X, Z\Omega^3_X) \to \mathrm{H}^1(X, \Omega^3_X)$ by abuse of notation. Since C is a vector space homomorphism of two spaces of the same dimension, it is injective if and only if it is surjective, thus $\mathrm{H}^1(X, B\Omega^3_X) = 0 \Leftrightarrow \mathrm{H}^2(X, B\Omega^3_X) = 0$ or equivalently $\mathrm{H}^2(X, B\Omega^2_X) = 0 \Leftrightarrow \mathrm{H}^3(X, B\Omega^2_X) = 0$ by Serre duality. It is then clear that (iii) or (iv) imply $\mathrm{H}^j(X, B\Omega^i_X) = 0$ for all pairs (i, j) so we get (i). □

Remark 3.10. Theorem A implies that if X is a cubic fourfold and the associated cone $C(X, L^m)$ is $(1, F)$-injective (where L is ample and $m \gg 0$), then X is skew 2-ordinary, so by Lemma 3.9 X is ordinary. In contrast, all cubic fourfolds X are trivially skew 1-ordinary by the proof of Lemma 3.9, so by Theorem A, $C(X, L^m)$ is F-injective for $m \gg 0$.

We conclude this section with some more observations on the interplay between our k-ordinary definitions and slopes of the crystalline cohomology of X. For example, Newton–Hodge k-ordinarity may be equivalently defined as follows:

Definition 3.11. *X is Newton-Hodge k ordinary in degree n if* $\operatorname{multslope}_i H^n_{crys}(X/W) = \operatorname{rk} H^{n-i}(\Omega^i_X)$ *for all $i < k$.*

Note that under this definition we need not assume X is Mazur–Ogus

Now recall that a smooth proper variety X is *Hodge–Witt* if $\mathrm{H}^j(X, W\Omega^i_{X/k})$ is a finitely generated $W(k)$-module for all i, j, where $W\Omega^\bullet_{X/k}$ denotes the de Rham–Witt complex.

Remark 3.12. Hodge–Witt varieties with torsionfree crystalline cohomology are Mazur–Ogus by [IR83, Théorème 4.7 pp.204].

Proposition 3.13. *Let X be Hodge–Witt and have torsionfree crystalline cohomology. Then the slope i_0-part of the n-th crystalline cohomology has dimension equal to $h^{i_0, n-i_0}$ if and only if*

$$\mathrm{H}^j(X, BW\Omega^i_X) \otimes K = 0$$

for $(i, j) = (i_0, n - i_0 + 1), (i_0 + 1, n - i_0)$.

In particular, under the above assumptions, X is Newton–Hodge k-ordinary in degree n iff

$$\mathrm{H}^j(X, BW\Omega^i_X) \otimes K = 0$$

for $i + j = n + 1, i \leq k$.

Proof. We have $\mathrm{H}^{i+j-1}_{\mathrm{crys}}(X/K)_{(i-1,i)} \simeq \mathrm{H}^j(X, BW\Omega^i_X) \otimes K$ by [CL98, Exposé III, Proposition 2.2]. The domino numbers are 0 by the degeneration of the slope spectral sequence which comes from Hodge–Witt assumption (see [Jos20, §2.23] for definition; the degeneration implies that the maps being used to construct the dominos are 0). Since X is Mazur–Ogus, by [Jos20, §2.24, §2.25.2], the slope numbers ([Jos20, §2.20]), Hodge numbers and Hodge–Witt numbers coincide. The slope number is defined by

$$m^{i,j} = \sum_{\lambda \in [i-1, i)} (\lambda - i + 1) \dim \mathrm{H}^{i+j}_{\mathrm{crys}}(X/K)_{[\lambda]} + \sum_{\lambda \in [i, i+1)} (i + 1 - \lambda) \dim \mathrm{H}^{i+j}_{\mathrm{crys}}(X/K)_{[\lambda]}$$

If the two cohomological vanishings hold, only the term with $\lambda = i_0$ in the summation of $m^{i_0, n-i_0}$ survives by $\mathrm{H}^{i+j-1}_{\mathrm{crys}}(X/K)_{(i-1,i)} \simeq \mathrm{H}^j(X, BW\Omega^i_X) \otimes K$. Hence,

$$h^{i_0, n-i_0} = m^{i_0, n-i_0} = \dim \mathrm{H}^n_{\mathrm{crys}}(X/K)_{[i_0]}$$

Conversely, if $h^{i_0,n-i_0} = \dim \mathrm{H}^n_{\mathrm{crys}}(X/K)_{[i_0]}$, since $m^{i_0,n-i_0}$ is also equal to this number,

$$\dim \mathrm{H}^n_{\mathrm{crys}}(X/K)_{[i_0]} = \sum_{\lambda\in[i-1,i)} (\lambda - i + 1) \dim \mathrm{H}^{i+j}_{\mathrm{crys}}(X/K)_{[\lambda]} + \sum_{\lambda\in[i,i+1)} (i + 1 - \lambda) \dim \mathrm{H}^{i+j}_{\mathrm{crys}}(X/K)_{[\lambda]}$$

Since all numbers in the summation are nonnegative, the multiplicity of the slope $\lambda \in (i_0 - 1, i_0 + 1) \setminus \{i_0\}$ is 0. By $\mathrm{H}^{i+j-1}_{\mathrm{crys}}(X/K)_{(i-1,i)} \simeq \mathrm{H}^j(X, BW\Omega^i_X) \otimes K$, we obtain the cohomological vanishings. □

4. The cone construction

In this section we discuss some properties of affine cone singularities, along with some relevant exact sequences for their sheaves of higher differential forms, which we will make use of during the proof of our main theorems.

4.1. **Cones over smooth varieties.** Let X be a smooth projective geometrically integral variety of dimension $d \geq 1$ over a perfect field k of characteristic $p > 0$ and $L \in \mathrm{Pic}(X)$ an ample line bundle.

Definition 4.1. *The* affine cone over X with conormal bundle L *is*

$$C(X, L) := \mathrm{Spec} \bigoplus_{m\geq 0} \mathrm{H}^0(X, L^m)$$

Note the section ring is finitely generated as L is ample. $C(X, L)$ is normal[2], and unless $(X, L) = (\mathbb{P}^d, \mathcal{O}(1))$ it has an isolated singularity at the vertex $v \in C(X, L)$ which corresponds to the irrelevant ideal $\bigoplus_{m>0} \mathrm{H}^0(X, L^m)$. Moreover $\dim(C(X, L)) = \dim X + 1 = d + 1$.

Assume the section ring is generated in degree 1. It turns out that the blowup

$$f : \widetilde{C} := \mathrm{Spec}_X \bigoplus_{m\geq 0} L^m = \mathrm{Bl}_v (C(X, L)) \longrightarrow C(X, L)$$

is a log resolution of $C(X, L)$, that is an isomorphism over $U := C(X, L) \setminus \{v\}$ and has reduced exceptional divisor $E \simeq X$; see [Kol13, §3.8] for details. Note $\widetilde{C}$ is the total space of L^{-1} and hence is naturally an $\mathbb{A}^1$-bundle over X; by restriction, we obtain a $\mathbb{G}_m$-torsor

$$\pi : U \simeq \mathrm{Spec}_X \bigoplus_{m\in\mathbb{Z}} L^m \to X \tag{18}$$

We refer to [Kol13, §3.1] and [KS21, Proposition B.3] for examples of singularities of $C(X, L)$ dependent on X and ω_X. We record some of these here as an indication of how 'bad' the singularity can get.

Example 4.2. $C(X, L)$ is a local complete intersection if and only if the embedding of X via L is a smooth complete intersection in $\mathbb{P}^n$. $C(\mathbb{P}^2, \mathcal{O}(3))$ is Gorenstein but not lci, $C(\mathbb{P}^1, \mathcal{O}(3))$ (i.e. the cone over the twisted cubic) is Cohen–Macaulay but not Gorenstein, and if X is an abelian variety of dimension ≥ 2, $C(X, \mathcal{O}(1))$ is not Cohen–Macaulay.

We now focus on the sheaves of higher differential forms on $C(X, L)$. Since it is affine, by the long exact sequence for cohomology with support

$$\ldots \to \mathrm{H}^j_v(C(X, L), \mathcal{F}) \to \mathrm{H}^j(C(X, L), \mathcal{F}) \to \mathrm{H}^j(U, \mathcal{F}) \to \ldots$$

[2]For example, $C(X, \mathcal{O}(1))$ is the normalization of the classical affine cone over X, defined as $V(\{f_i\}) \subset \mathbb{A}^{n+1}$ when X is defined by the homogeneous equations $f_i = 0$ inside $\mathbb{P}^n$.

we observe the following after substituting $\mathcal{F} = \Omega^i_{C(X,L)}$ and $\Omega^{[i]}_{C(X,L)}$:

Lemma 4.3. *For $j \geq 2$ and $d \geq 1$ we have isomorphisms*

$$\mathrm{H}^j_v(C(X,L), \Omega^i_{C(X,L)}) \simeq \mathrm{H}^j_v(C(X,L), \Omega^{[i]}_{C(X,L)}) \simeq \mathrm{H}^{j-1}(U, \Omega^i_U)$$

In lower degrees we have an exact sequence

$$0 \to \mathrm{H}^0_v(C(X,L), \Omega^i_{C(X,L)}) \to \Omega^i_{C(X,L)} \to \Omega^{[i]}_{C(X,L)} \to \mathrm{H}^1_v(C(X,L), \Omega^i_{C(X,L)}) \to 0$$

after identifying coherent sheaves on affine schemes with their underlying modules.

Note that by definition $\Omega^{[i]}_{C(X,L)}$ are reflexive and so $\mathrm{H}^j_v(C(X,L), \Omega^{[i]}_{C(X,L)}) = 0$ for $j < 2$, while the difference with $\Omega^i_{C(X,L)}$ in degrees 0 and 1 measures how far $\Omega^i_{C(X,L)} \to \Omega^{[i]}_{C(X,L)}$ is from being an isomorphism. The same conclusion of Lemma 4.3 applies to sheaves $Z\Omega^i_{C(X,L)}, B\Omega^i_{C(X,L)}$ and their reflexivizations as well.

4.2. **Sequences of differential forms in characteristic p.** Retaining the notation of Section 4.1, let $\pi : U \to X$ be the $\mathbb{G}_m$-torsor associated with the cone $C(X,L)$. Since it is a smooth morphism, the relative cotangent sequence and its higher differential variant read

$$0 \to \pi^*\Omega^1_X \to \Omega^1_U \to \mathcal{O}_U \to 0 \tag{19}$$

$$0 \to \pi^*\Omega^i_X \to \Omega^i_U \to \pi^*\Omega^{i-1}_X \to 0 \tag{20}$$

where we identify $\Omega^1_{U/X} \simeq \mathcal{O}_U$ and we have by convention $\Omega^{<0}_X = 0$; (20) follows from (19) by taking exterior powers.

Since π is affine, π_* is exact, and a local calculation yields $\pi_*\mathcal{O}_U \simeq \bigoplus_m L^m$. Therefore pushing forward (20) to X we obtain

$$0 \to \bigoplus_{m\in\mathbb{Z}} \Omega^i_X \otimes L^m \to \pi_*\Omega^i_U \xrightarrow{\mathrm{res}} \bigoplus_{m\in\mathbb{Z}} \Omega^{i-1}_X \otimes L^m \to 0$$

which we will reprove below (cf. Proposition 4.5) in coordinates. We first describe the residue map $\mathrm{res} : \pi_*\Omega^i_U \to \bigoplus_{m\in\mathbb{Z}} \Omega^{i-1}_X \otimes L^m$ as follows.

By choosing an affine open subvariety $V = \mathrm{Spec}(R) \subset X$ where the $\mathbb{G}_m$-fibration $\pi : U \to X$ trivializes, one can fix an isomorphism $\pi^{-1}(V) \simeq V \times_k \mathrm{Spec}\, k[T^{\pm 1}] \simeq \mathrm{Spec}\, R[T^{\pm 1}]$, whence

$$\Omega^i_U(\pi^{-1}(V)) \simeq \Omega^i_{R[T^{\pm1}]/k} \simeq (R[T^{\pm1}] \otimes_R \Omega^i_{R/k}) \oplus (R[T^{\pm1}] \otimes_R \Omega^{i-1}_{R/k} \wedge dT) \tag{21}$$

Thus, local sections $\Gamma(\pi^{-1}(V), \Omega^i_U)$ are generated by forms $\omega = T^a\omega_1 + T^b dT \wedge \omega_2$ where $\omega_1 \in \Omega^i_X(V)$, $\omega_2 \in \Omega^{i-1}_X(V)$, and $\mathrm{res}(\omega) = \omega_2 \otimes T^{b+1} \in (\bigoplus_m \Omega^{i-1}_X \otimes L^m)(V)$.

We next describe a natural $\mathbb{Z}$-grading on sheaves $\pi_*\Omega^i_U$ via the $\mathbb{G}_m$-torsor $\pi : U \to X$. By choosing affine open $V = \mathrm{Spec}(R) \subset X$ so that we obtain a trivialization $\mathrm{Spec}\, R[T^{\pm1}] \simeq U \times_X V \to V$ of the torsor as above, we declare $T, dT \in \pi_*\Omega^\bullet_U(V)$ to have grading 1, and any horizontal differential form to have grading 0.

It is clear that this local description is well-defined, independent of the trivialization and it globalizes to give a sub-$\mathcal{O}_X$-module $(\pi_*\Omega^i_U)_n \subset \pi_*\Omega^i_U$ generated locally by sections of the form $\omega = T^n\omega_1 + T^{n-1}dT \wedge \omega_2$, where $\omega_1 \in \Omega^i_X(V)$ and $\omega_2 \in \Omega^{i-1}_X(V)$.

Remark 4.4. An equivalent formulation when k is algebraically closed is given by writing U as the relative spectrum $\mathrm{Spec}_X(\pi_*\mathcal{O}_U)$ as in (18). The $\mathcal{O}_X$-algebra morphism $\pi_*\mathcal{O}_U \to \pi_*\mathcal{O}_U$ given by multiplication by a nonzero scalar λ^n on component L^n gives rise to a unique automorphism $\alpha_\lambda : U \to U$ so that $\pi \circ \alpha_\lambda = \pi$. The induced map $\alpha_\lambda^\# : \pi_*\Omega^i_U \to \pi_*\Omega^i_U$ gives a description of the local sections of $(\pi_*\Omega^i_U)_n$ as

$$(\pi_*\Omega^i_U)_n(V) = \{\omega \in \pi_*\Omega^i_U(V) \mid \alpha_\lambda^\#(\omega) = \lambda^n\omega \text{ for all } \lambda \in k^\times\}$$

on affine opens $V \subset X$. From this description we see that T, dT have degree 1 and as α_λ is a morphism over X, the horizontal differentials (i.e. the differentials coming from X) have degree 0.

Proposition 4.5. *For each $i \geq 0$, we have a short exact sequence of graded $\mathcal{O}_X$-modules:*

$$0 \to \bigoplus_{m\in\mathbb{Z}} \Omega^i_X \otimes L^m \to \pi_*\Omega^i_U \xrightarrow{\text{res}} \bigoplus_{m\in\mathbb{Z}} \Omega^{i-1}_X \otimes L^m \to 0 \tag{22}$$

In particular, we have short exact sequences of graded pieces for each integer n:

$$0 \to \Omega^i_X \otimes L^n \to (\pi_*\Omega^i_U)_n \to \Omega^{i-1}_X \otimes L^n \to 0 \tag{23}$$

Proof. Choose an affine open $V = \mathrm{Spec}(R) \subset X$ and a trivialization $U \times_X V \simeq \mathrm{Spec}\, R[T^{\pm 1}]$ as above. By the local sections description in (21), the left map is defined by sending $\omega \otimes T^a \mapsto T^a\omega$ for $\omega \in \Omega^i_X(V)$, and the residue map sends $T^a\omega_1 + T^b dT \wedge \omega_2 \mapsto \omega_2 \otimes T^{b+1}$ for $\omega_1 \in \Omega^i_X(V)$, $\omega_2 \in \Omega^{i-1}_X(V)$. We thus see that res is surjective and its kernel is locally generated by sections of form $T^a\omega$, yielding exact sequence (22).

We also obtain the graded version in exact sequence (23) by noting that a local section of $(\pi_*\Omega^i_U)_n$ of the form $T^n\omega_1 + T^{n-1}dT \wedge \omega_2$ maps through the residue map to $\omega_2 \otimes T^n$ and the kernel of the residue map is generated by $\omega_1 \otimes T^n$. □

By exactness, Proposition 4.5 remains true after pushing forward along the absolute Frobenius F_X. Recall the definition of cocycles $Z\Omega^i_X \subset F_*\Omega^i_X$ and coboundaries $B\Omega^i_X \subset F_*\Omega^i_X$ from (1) and (2), respectively; we will freely denote the absolute Frobenii on X and U by F. The differentials of the complex $F_*\Omega^\bullet_U$ are compatible with the $\mathbb{Z}$-grading; more precisely, $\pi_* d_U : \pi_*\Omega^i_U \to \pi_*\Omega^{i+1}_U$ commutes with $\alpha_\lambda^\#$ in the notation of Remark 4.4, and F_* commutes with both maps. Thus both $\pi_*Z\Omega^i_U$, $\pi_*B\Omega^i_U$ inherit $\mathbb{Z}$-gradings from $F_*\pi_*\Omega^i_U$ as follows:

$$(\pi_*Z\Omega^i_U)_n := \ker(F_*(\pi_*\Omega^i_U)_n \to F_*(\pi_*\Omega^{i+1}_U)_n) \tag{24}$$

$$(\pi_*B\Omega^i_U)_n := \mathrm{im}(F_*(\pi_*\Omega^{i-1}_U)_n \to F_*(\pi_*\Omega^i_U)_n) \tag{25}$$

We can now prove analogues of Proposition 4.5 for $B\Omega^i_U$ and $Z\Omega^i_U$.

Lemma 4.6. *For each $i \geq 0$, we have a short exact sequence of graded $\mathcal{O}_X$-modules*

$$0 \to \bigoplus_{p|m} B\Omega^i_X \otimes L^{m/p} \to \pi_*B\Omega^i_U \xrightarrow{\text{res}} \begin{matrix} \oplus_{p|m} B\Omega^{i-1}_X \otimes L^{m/p} \\ \oplus \\ \oplus_{p\nmid m} F_*\left(\Omega^{i-1}_X \otimes L^m\right) \end{matrix} \to 0$$

Proof. We construct the right morphism by taking the Frobenius pushforward of the residue map $\pi_*\Omega^i_U \xrightarrow{\text{res}} \bigoplus_{m\in\mathbb{Z}} \Omega^{i-1}_X \otimes L^m$ and then restricting the domain to $\pi_*B\Omega^i_U$. We fix, as before, a local trivialization of $\mathbb{G}_m$-torsor U via open affine $V = \mathrm{Spec}(R) \subset X$ so that $U \times_X V \simeq \mathrm{Spec}\, R[T^{\pm 1}]$. Following the description of local sections of $\pi_*\Omega^i_U$ in (21), we set

$t := F_*T \in F_*(R[T^{\pm 1}])$ for the Frobenius-twisted local parameter of $\mathbb{G}_m$, so $dt = F_*(dT)$ and $t^k F_*\omega = F_*(T^k\omega)$. The local sections of $\pi_* B\Omega^i_U$ are therefore generated by expressions of the form

$$d(t^a\omega_1) = at^{a-1}dt\wedge\omega_1 + t^a d\omega_1, \quad d(t^b dt\wedge\omega_2) = -t^b dt\wedge d\omega_2, \quad \omega_1 \in F_*\Omega^{i-1}_X(V), \omega_2 \in F_*\Omega^{i-2}_X(V)$$

which the residue map respectively sends to

$$a\omega_1 \otimes t^a, \quad -d\omega_2 \otimes t^{b+1}$$

We write $\mathrm{res} = \bigoplus_{N\in\mathbb{Z}} \mathrm{res}_N$ where

$$\mathrm{res}_N : \pi_* B\Omega^i_U \hookrightarrow \pi_* F_*\Omega^i_U \xrightarrow{\mathrm{res}} \bigoplus_{m\in\mathbb{Z}} F_*(\Omega^{i-1}_X \otimes L^m) \twoheadrightarrow F_*(\Omega^{i-1}_X \otimes L^N)$$

and we note that res_N is surjective when $p \nmid N$ in view of the expression $a\omega_1 \otimes t^a$, while when $p \mid N$, the image of res_N is $B\Omega^{i-1}_X \otimes L^{N/p}$. For the kernel of the residue map we may reindex a local section of $\pi_* B\Omega^i_U$ as

$$\phi = d(\sum_m (t^m\omega_m + t^{m-1}dt\wedge\eta_m)) = \sum_m (mt^{m-1}dt\wedge\omega_m + t^m d\omega_m - t^{m-1}dt\wedge d\eta_m)$$

where $\omega_m \in F_*\Omega^{i-1}_X(V), \eta_m \in F_*\Omega^{i-2}_X(V)$. Then $\mathrm{res}(\phi) = 0$ implies $\sum_m (mt^m\omega_m - t^m d\eta_m) = 0$, i.e. η_m is a cocycle when $p|m$, and $\omega_m = m^{-1}d\eta_m$ when $p \nmid m$. Thus

$$\phi = \sum_m t^m d\omega_m = \sum_{p|m} t^m d\omega_m$$

and so the kernel is identified with $\bigoplus_{p|m} B\Omega^i_X \otimes L^{m/p}$ inside $\bigoplus_{p|m} F_*(\Omega^i_X \otimes L^m)$. □

Corollary 4.6.1. *For each integer n we have short exact sequences*

$$0 \to B\Omega^i_X \otimes L^n \to (\pi_* B\Omega^i_U)_{pn} \to B\Omega^{i-1}_X \otimes L^n \to 0 \tag{26}$$

$$0 \to (\pi_* B\Omega^i_U)_n \to F_*(\pi_*\Omega^i_U)_n \to \left(\pi_*\left(\frac{F_*\Omega^i_U}{B\Omega^i_U}\right)\right)_n \to 0 \tag{27}$$

Proof. The first exact sequence follows directly from the definition of $(\pi_* B\Omega^i_U)_{pn}$ in (25) and the description of local sections in the proof of Lemma 4.6. Explicitly, for an open affine $V = \mathrm{Spec}(R) \subset X$ so that $U \times_X V \simeq \mathrm{Spec}\, R[T^{\pm 1}]$, a local section has form

$$t^{pn}d\omega_1 + t^{pn-1}dt\wedge d\omega_2$$

where $\omega_1 \in F_*\Omega^{i-1}_X(V), \omega_2 \in F_*\Omega^{i-2}_X(V)$. Then the residue map sends this section to $d\omega_2 \otimes t^{pn} \in B\Omega^{i-1}_X \otimes L^n \subset F_*(\Omega^{i-1}_X \otimes L^{pn})$, and the kernel of the residue map is identified with $B\Omega^i_X \otimes L^n$. The second exact sequence follows directly from the inclusion $\pi_* B\Omega^i_U \hookrightarrow \pi_* F_*\Omega^i_U$ respecting the grading, upon identifying $\left(\pi_*\left(\frac{F_*\Omega^i_U}{B\Omega^i_U}\right)\right)_n = \frac{F_*(\pi_*\Omega^i_U)_n}{(\pi_* B\Omega^i_U)_n}$. □

By construction, the short exact sequence of the preceding lemma fits in the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{p|m} B\Omega_X^i \otimes L^{m/p} & \longrightarrow & \pi_* B\Omega_U^i & \xrightarrow{\text{res}} & \begin{array}{c}\bigoplus_{p|m} B\Omega_X^{i-1}\otimes L^{m/p} \\ \oplus \\ \bigoplus_{p\nmid m} F_*\left(\Omega_X^{i-1}\otimes L^m\right)\end{array} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_{m\in\mathbb{Z}} F_*(\Omega_X^i \otimes L^m) & \longrightarrow & \pi_* F_*\Omega_U^i & \xrightarrow{\text{res}} & \bigoplus_{m\in\mathbb{Z}} F_*(\Omega_X^{i-1} \otimes L^m) & \longrightarrow & 0 \end{array}$$

By the Snake Lemma, we obtain the corresponding short exact sequence and its graded counterpart on the quotients:

Lemma 4.7. *For each $i \geq 0$, we have a short exact sequence of graded $\mathcal{O}_X$-modules, where the left and right terms are graded by the integers $m \in \mathbb{Z}$.*

$$0 \to \begin{array}{c}\bigoplus_{p|m} \frac{F_*\Omega_X^i}{B\Omega_X^i}\otimes L^{m/p} \\ \oplus \\ \bigoplus_{p\nmid m} F_*\left(\Omega_X^i\otimes L^m\right)\end{array} \to \pi_* \frac{F_*\Omega_U^i}{B\Omega_U^i} \to \bigoplus_{p|m} \frac{F_*\Omega_X^{i-1}}{B\Omega_X^{i-1}} \otimes L^{m/p} \to 0$$

In particular, we have a short exact sequence on the graded pieces for each integer n:

$$0 \to \frac{F_*\Omega_X^i}{B\Omega_X^i} \otimes L^n \to \left(\pi_*\left(\frac{F_*\Omega_U^i}{B\Omega_U^i}\right)\right)_{pn} \to \frac{F_*\Omega_X^{i-1}}{B\Omega_X^{i-1}} \otimes L^n \to 0$$

We next construct a short exact sequence for the cocycles $\pi_* Z\Omega_U^i$.

Lemma 4.8. *For each $i \geq 0$, we have a short exact sequence of graded $\mathcal{O}_X$-modules, where the left and right terms are graded by the integers $m \in \mathbb{Z}$.*

$$0 \to \bigoplus_{p|m} Z\Omega_X^i \otimes L^{m/p} \to \pi_* Z\Omega_U^i \xrightarrow{\text{res}} \begin{array}{c}\bigoplus_{p|m} Z\Omega_X^{i-1}\otimes L^{m/p} \\ \oplus \\ \bigoplus_{p\nmid m} F_*\left(\Omega_X^{i-1}\otimes L^m\right)\end{array} \to 0$$

In particular, we have a short exact sequence on the graded pieces for each n:

$$0 \to Z\Omega_X^i \otimes L^n \to (\pi_* Z\Omega_U^i)_{pn} \to Z\Omega_X^{i-1} \otimes L^n \to 0$$

Proof. We work analogously to Lemma 4.6 with $Z\Omega_U^i$ instead of $B\Omega_U^i$; once again we choose a local trivialization of $\mathbb{G}_m$-torsor $\pi : U \to X$ and, using the notation in the proof of Lemma 4.6, we write a local section of $\pi_* Z\Omega_U^i$ as $\sum_m (t^m\omega_m + t^{m-1}dt \wedge \eta_m)$ for some local sections ω_m of $F_*\Omega_X^i$ and η_m of $F_*\Omega_X^{i-1}$. Since

$$d(\sum_m (t^m\omega_m + t^{m-1}dt \wedge \eta_m)) = \sum_m (mt^{m-1}dt \wedge \omega_m + t^m d\omega_m - t^{m-1}dt \wedge d\eta_m)$$

we see that ω_m is a cocycle and $m\omega_m - d\eta_m = 0$ for each m. When $p|m$, the latter condition says that η_m is a cocycle. When $p \nmid m$, η_m can be an arbitrary differential form, and $\omega_m = m^{-1}d\eta_m$ (in which case ω_m is automatically a cocycle).

The image of $\sum_m (t^m\omega_m + t^{m-1}dt \wedge \eta_m)$ via the residue map is $\sum_m \eta_m \otimes t^m$, so the image of $\pi_* Z\Omega_U^i$ via the residue map is $\left(\bigoplus_{p|m} Z\Omega_X^{i-1} \otimes L^{m/p}\right) \oplus \bigoplus_{p\nmid m} F_*\left(\Omega_X^{i-1} \otimes L^m\right)$ due to the conditions on η_m above.

The kernel of $\text{res}|_{\pi_* Z\Omega_U^i}$ consists of the elements $\sum_m (t^m\omega_m + t^{m-1}dt \wedge \eta_m)$ which satisfy the above conditions as well as the vanishing $\sum_m \eta_m \otimes t^m = 0$. It immediately follows that this

kernel is $\bigoplus_{p|m} Z\Omega^i_X \otimes L^{m/p}$. The graded version of the exact sequence then follows exactly as in the proof of Corollary 4.6.1. □

Applying the Snake Lemma to the following commutative diagram with exact rows, we obtain Lemma 4.9 below.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{p|m} B\Omega^i_X \otimes L^{m/p} & \longrightarrow & \pi_* B\Omega^i_U & \xrightarrow{\text{res}} & \begin{array}{c}\bigoplus_{p|m} B\Omega^{i-1}_X \otimes L^{m/p} \\ \oplus \\ \bigoplus_{p\nmid m} F_*\left(\Omega^{i-1}_X \otimes L^m\right)\end{array} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{p|m} Z\Omega^i_X \otimes L^{m/p} & \longrightarrow & \pi_* Z\Omega^i_U & \xrightarrow{\text{res}} & \begin{array}{c}\bigoplus_{p|m} Z\Omega^{i-1}_X \otimes L^{m/p} \\ \oplus \\ \bigoplus_{p\nmid m} F_*\left(\Omega^{i-1}_X \otimes L^m\right)\end{array} & \longrightarrow & 0
\end{array}$$

Lemma 4.9. *For each $i \geq 0$, we have a short exact sequence of graded $\mathcal{O}_X$-modules, where the left and right terms are graded by the integers $m \in \mathbb{Z}$.*

$$0 \to \bigoplus_{p|m} \frac{Z\Omega^i_X}{B\Omega^i_X} \otimes L^{m/p} \to \pi_* \frac{Z\Omega^i_U}{B\Omega^i_U} \to \bigoplus_{p|m} \frac{Z\Omega^{i-1}_X}{B\Omega^{i-1}_X} \otimes L^{m/p} \to 0$$

In particular, we have a short exact sequence on the graded pieces for each n:

$$0 \to \frac{Z\Omega^i_X}{B\Omega^i_X} \otimes L^n \to \left(\pi_* \frac{Z\Omega^i_U}{B\Omega^i_U}\right)_{pn} \to \frac{Z\Omega^{i-1}_X}{B\Omega^{i-1}_X} \otimes L^n \to 0$$

Note this graded exact sequence is consistent with that of Proposition 4.5 because the Cartier isomorphism $Z\Omega^i_U/B\Omega^i_U \simeq \Omega^i_U$, after applying π_*, maps graded components of weight pn to graded components of weight n via

$$\left(\pi_* \left(\frac{Z\Omega^i_U}{B\Omega^i_U}\right)\right)_{pn} \xrightarrow{\sim} (\pi_* \Omega^i_U)_n \tag{28}$$

By Lemma 4.7, Lemma 4.9 and yet another application of the Snake Lemma, we have a commutative diagram of $\mathbb{Z}$-graded $\mathcal{O}_X$-modules with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{p|m} \frac{Z\Omega^i_X}{B\Omega^i_X} \otimes L^{m/p} & \longrightarrow & \pi_* \frac{Z\Omega^i_U}{B\Omega^i_U} & \longrightarrow & \bigoplus_{p|m} \frac{Z\Omega^{i-1}_X}{B\Omega^{i-1}_X} \otimes L^{m/p} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \begin{array}{c}\bigoplus_{p|m} \frac{F_*\Omega^i_X}{B\Omega^i_X} \otimes L^{m/p} \\ \oplus \\ \bigoplus_{p\nmid m} F_*\left(\Omega^i_X \otimes L^m\right)\end{array} & \longrightarrow & \pi_* \frac{F_*\Omega^i_U}{B\Omega^i_U} & \longrightarrow & \bigoplus_{p|m} \frac{F_*\Omega^{i-1}_X}{B\Omega^{i-1}_X} \otimes L^{m/p} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \begin{array}{c}\bigoplus_{p|m} \frac{F_*\Omega^i_X}{Z\Omega^i_X} \otimes L^{m/p} \\ \oplus \\ \bigoplus_{p\nmid m} F_*\left(\Omega^i_X \otimes L^m\right)\end{array} & \longrightarrow & \pi_* \frac{F_*\Omega^i_U}{Z\Omega^i_U} & \longrightarrow & \bigoplus_{p|m} \frac{F_*\Omega^{i-1}_X}{Z\Omega^{i-1}_X} \otimes L^{m/p} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We end this section with some observations on the weight subsheaves $(\pi_*\Omega^i_U)_n$ and surjectivity of the reflexive Cartier operator, which will be helpful later.

Lemma 4.10. *Under our $\mathbb{Z}$-grading convention, $(\pi_*\Omega^i_U)_n \simeq (\pi_*\Omega^i_U)_0 \otimes L^n$ for all n.*

Proof. We have a natural map $(\pi_*\Omega^i_U)_0 \otimes L^n \to (\pi_*\Omega^i_U)_n$ sending $\omega \otimes \lambda \mapsto \lambda\omega$. Once again, work locally and trivialize L over an open affine $V = \operatorname{Spec}(R)$ so that $U \times_X V \simeq \operatorname{Spec} R[T^{\pm 1}]$. The map becomes $\omega_1 + dT/T \wedge \omega_2 \mapsto T^n\omega_1 + T^{n-1}dT \wedge \omega_2$ i.e. it is multiplication by T^n, with obvious inverse T^{-n}. We therefore get an isomorphism of sheaves. □

Similar statements hold for $\mathcal{M} = B\Omega^i_U, Z\Omega^i_U$ and $F_*\Omega^i_U/B\Omega^i_U$; via the projection formula and multiplication by T^{pk}, we uniformly get

$$(\pi_*\mathcal{M})_{pk+\ell} \simeq (\pi_*\mathcal{M})_\ell \otimes L^k$$

Lemma 4.11. *Suppose $\mathrm{H}^1(X, B\Omega^i_X \otimes L^m) = 0$ for $i \leq k$ and all m, then the reflexivized Cartier operator*

$$C : Z\Omega^{[i]}_{C(X,L)} \to \Omega^{[i]}_{C(X,L)}$$

is surjective for $i \leq k$.

Proof. Write $\Omega^{[i]}_{C(X,L)} = j_*\Omega^i_U$, $Z\Omega^{[i]}_{C(X,L)} = j_*Z\Omega^i_U$ for $j : U \hookrightarrow C(X,L)$ the inclusion of the punctured cone. We have exact sequence

$$0 \to j_*B\Omega^i_U \to j_*Z\Omega^i_U \xrightarrow{j_*C_U} j_*\Omega^i_U \to \mathrm{R}^1 j_*B\Omega^i_U \to \mathrm{R}^1 j_*Z\Omega^i_U$$

Thus j_*C_U is surjective if and only if $\mathrm{R}^1 j_*B\Omega^i_U \to \mathrm{R}^1 j_*Z\Omega^i_U$ is injective. Note that the Leray spectral sequence $E_2^{pq} = \mathrm{H}^p(C(X,L), \mathrm{R}^q j_*B\Omega^i_U) \Rightarrow \mathrm{H}^{p+q}(U, B\Omega^i_U)$ degenerates at E_2 as $E_2^{p,q} = 0$ for $p > 0$ by affineness of $C(X,L)$, whence $\mathrm{H}^1(U, B\Omega^i_U) \simeq \mathrm{H}^0(C(X,L), \mathrm{R}^1 j_*B\Omega^i_U)$ and similarly for $\mathrm{R}^1 j_*Z\Omega^i_U$. Hence $\mathrm{R}^1 j_*B\Omega^i_U \to \mathrm{R}^1 j_*Z\Omega^i_U$ is injective if and only if $\mathrm{H}^1(U, B\Omega^i_U) \to \mathrm{H}^1(U, Z\Omega^i_U)$ is injective, on account of $C(X,L)$ being affine again.

Applying exact functor π_* and decomposing by weights, it suffices to show

$$\mathrm{H}^1(X, (\pi_*B\Omega^i_U)_m) \to \mathrm{H}^1(X, (\pi_*Z\Omega^i_U)_m) \tag{29}$$

is injective for all m. When $p \nmid m$, Lemma 4.6 and Lemma 4.8 yield

$$(\pi_*B\Omega^i_U)_m \simeq (\pi_*Z\Omega^i_U)_m \simeq F_*(\Omega^{i-1}_X \otimes L^m)$$

so there is nothing to prove. When $m = pn$ we use Corollary 4.6.1 to bound

$$\cdots \to \mathrm{H}^1(X, B\Omega^i_X \otimes L^n) \to \mathrm{H}^1(X, (\pi_*B\Omega^i_U)_{pn}) \to \mathrm{H}^1(X, B\Omega^{i-1}_X \otimes L^n) \to \cdots$$

So $\mathrm{H}^1(X, B\Omega^i_X \otimes L^m) = 0$ for $i \leq k$ and all m makes (29) automatically injective. □

A byproduct of the above is a weaker assumption that still implies Cartier surjectivity; it is however not as clean to state and check as Lemma 4.11.

Corollary 4.11.1. *If $\mathrm{H}^1(X, (\pi_*B\Omega^i_U)_0 \otimes L^m) = 0$ for $i \leq k$ and all m, then the reflexivized Cartier operator $C : Z\Omega^{[i]}_{C(X,L)} \to \Omega^{[i]}_{C(X,L)}$ is surjective for $i \leq k$.*

5. (k, F)-injectivity for cones

In this section we prove Theorem A. We recall the setting of Definition 4.1 and we let X be a smooth projective variety of dimension $d \geq 1$ over a perfect field k of characteristic p, L an ample line bundle on X and $C(X, L)$ the affine cone over X with conormal bundle L. Our goal is to relate the (k, F)-injectivity of $C(X, L)$ with certain arithmetic and topological properties of X; these conditions will amount to X being ordinary in one of the ways described in Section 3 (see Definition 3.4) and its higher differential sheaves $\Omega^i_X, B\Omega^i_X$ satisfying some cohomological vanishing after twisting by L. The main results are summarized in Theorem 5.9 and Corollary 5.7.1.

We work by inducting on k so we start with Lemma 5.1 below as our base case. Note that the statement itself is commonly known to experts, and a similar argument appears in [BST17, Proof of Thm. 4.2] for L sufficiently ample. We supply a more general proof here that is better adapted to the higher k considerations as well.

Before the statement, we will also need the notion of Frobenius-stable sections. Given a line bundle $\mathcal{F}$, the natural trace map $\mathrm{Tr} : F_*\omega_X \to \omega_X$ induces for each m a map

$$\mathrm{Tr}_m(\mathcal{F}) : \mathrm{H}^0(X, F^m_*(\omega_X \otimes \mathcal{F}^{p^m})) \longrightarrow \mathrm{H}^0(X, \omega_X \otimes \mathcal{F}) \tag{30}$$

by iterating the trace. We define the Frobenius-stable sections of $\omega_X \otimes \mathcal{F}$ as

$$S^0(\omega_X \otimes \mathcal{F}) := \bigcap_{m \geq 0} \mathrm{Im}\,(\mathrm{Tr}_m(\mathcal{F})) \tag{31}$$

The images of Tr_m form a stabilizing descending chain and so $S^0(\omega_X \otimes \mathcal{F})$ is a well-defined subspace of $\mathrm{H}^0(X, \omega_X \otimes \mathcal{F})$.

Lemma 5.1 (The $k = 0$ case)**.** *Let X, L and $C(X, L)$ be as above. Then $C(X, L)$ is F-injective if and only if*

(i) *X is Bloch–Kato 1-ordinary*
(ii) *$\mathrm{H}^i(X, L^m) = \mathrm{H}^i(X, \omega_X \otimes L^m) = 0$ for $m, i > 0$*
(iii) *$S^0(\omega_X \otimes L^m) = \mathrm{H}^0(X, \omega_X \otimes L^m)$ for $m > 0$*

Proof. We first note that X being Bloch–Kato 1-ordinary is equivalent to

$$F^* : \mathrm{H}^i(X, \mathcal{O}_X) \xrightarrow{\sim} \mathrm{H}^i(X, F_*\mathcal{O}_X) \text{ for all } i$$

by Remark 3.5 (i). We also recall the setting of Section 4.1 and set U to be the punctured cone, with associated $\mathbb{G}_m$-torsor structure $\pi : U \to X$. Then Lemma 4.3 yields $\mathrm{H}^{i+1}_{\mathfrak{m}}(\mathcal{O}_{C(X,L)}) \simeq \mathrm{H}^i(U, \mathcal{O}_U)$ for $i \geq 1$; note $\mathrm{H}^i_{\mathfrak{m}}(\mathcal{O}_{C(X,L)}) = 0$ for $i \leq 1$ as $C(X, L)$ is normal. Together with $\pi_*\mathcal{O}_U \simeq \bigoplus_{m \in \mathbb{Z}} L^m$ we obtain

$$\mathrm{H}^{i+1}_{\mathfrak{m}}(\mathcal{O}_{C(X,L)}) \simeq \mathrm{H}^i(U, \mathcal{O}_U) \simeq \mathrm{H}^i(X, \pi_*\mathcal{O}_U) \simeq \bigoplus_{m \in \mathbb{Z}} \mathrm{H}^i(X, L^m), \tag{32}$$

$$\mathrm{H}^{i+1}_{\mathfrak{m}}(F_*\mathcal{O}_{C(X,L)}) \simeq \mathrm{H}^i(X, F_*\pi_*\mathcal{O}_U) \simeq \bigoplus_{m \in \mathbb{Z}} \mathrm{H}^i(X, L^m) \tag{33}$$

The Frobenius $\mathcal{O}_{C(X,L)} \to F_*\mathcal{O}_{C(X,L)}$ induces the Frobenius on $U = \mathrm{Spec}_X\,(\bigoplus_m L^m)$ which corresponds to an $\mathcal{O}_X$-algebra morphism

$$\bigoplus_{m \in \mathbb{Z}} L^m \to \bigoplus_{m \in \mathbb{Z}} F_*(L^m)$$

so that component L^m maps to component $F_*(L^{mp})$. We thus obtain from (32), (33) the following Frobenius-semilinear map of graded $\bigoplus_{n\geq 0}\mathrm{H}^0(X,L^n)$-modules, which decomposes into pieces:

$$\alpha=\bigoplus_{m\in\mathbb{Z}}\alpha_m:\mathrm{H}^{i+1}_{\mathfrak{m}}(\mathcal{O}_{C(X,L)})\to\mathrm{H}^{i+1}_{\mathfrak{m}}(F_*\mathcal{O}_{C(X,L)})$$

$$\alpha_m:\mathrm{H}^i(X,L^m)\to\mathrm{H}^i(X,F_*(L^{mp}))\simeq\mathrm{H}^i(X,L^{mp})$$

Then $C(X,L)$ is F-injective if and only if α is injective for all $i+1\leq d+1$, which holds if and only if α_m is injective for all $m\in\mathbb{Z}$ and $i\leq d$.

Now suppose X satisfies conditions (i)-(iii). Map α_m is injective (in fact bijective) for $m=0$ by X being Bloch–Kato 1-ordinary. Condition (ii) makes α_m trivially injective for $m>0$, while for $m<0$ we have $\mathrm{H}^i(X,L^m)\simeq\mathrm{H}^{d-i}(X,\omega_X\otimes L^{-m})^\vee$ by Serre duality, which vanishes by (ii) for $m<0, i<d$. So it remains to check α_m is injective when $i=d$ and $m<0$. In this case $\mathrm{H}^d(X,L^m)\to\mathrm{H}^d(X,L^{mp})$ is injective if and only if the dual map

$$\mathrm{Tr}:\mathrm{H}^0(X,F_*(\omega_X\otimes L^{-mp}))\to\mathrm{H}^0(X,\omega_X\otimes L^{-m})$$

is surjective. Note that Tr is surjective for all $m<0$ if and only if $\mathrm{Tr}_n(L^{-m})$ is surjective for all $n\geq 1, m<0$ by (30), if and only if $S^0(\omega_X\otimes L^{-m})=\mathrm{H}^0(X,\omega_X\otimes L^{-m})$ by (31), which is (iii). Altogether, conditions (i)-(iii) imply $C(X,L)$ is F-injective.

For the converse, F-injectivity of $C(X,L)$ is equivalent to all α_m being injective, as before. For $m<0, i=d$ we showed that injectivity is equivalent to condition (iii), while for $m=0$ (and any i) we showed injectivity is equivalent to X being Bloch–Kato 1-ordinary. For the remaining cases, note that when $m>0$ we iterate α_m to get

$$\mathrm{H}^i(X,L^m)\hookrightarrow\mathrm{H}^i(X,L^{mp^N})$$

and the target is zero for $i>0, N\gg 0$ by Serre vanishing. When $m<0$ and $0<i<d$, by Serre duality we have surjections $\alpha_m^\vee:\mathrm{H}^{d-i}(X,\omega_X\otimes L^{-mp})\twoheadrightarrow\mathrm{H}^{d-i}(X,\omega_X\otimes L^{-m})$, and so via iteration

$$\mathrm{H}^{d-i}(X,\omega_X\otimes L^{-mp^N})\twoheadrightarrow\mathrm{H}^{d-i}(X,\omega_X\otimes L^{-m})$$

As the source is zero for $m<0, N\gg 0$ by Serre vanishing, so is $\mathrm{H}^{d-i}(X,\omega_X\otimes L^{-m})$ and we obtain (ii); note that $\mathrm{H}^d(X,\omega_X\otimes L^{-m})\simeq\mathrm{H}^0(X,L^m)^\vee=0$ automatically. □

Remark 5.2. Some observations on the statement of the lemma:

(i) The analogous criterion in characteristic 0 says that $C(X,L)$ is Du Bois if and only if $\mathrm{H}^i(X,L^m)=0$ for $i,m>0$ ([KS21, Prop. B3]). Though F-injective is viewed as a characteristic p analogue of the Du Bois property, we see that the characteristic p criterion here is more involved. In particular, (i) and (iii) come from Frobenius while $\mathrm{H}^i(X,\omega_X\otimes L^m)=0$ in (ii) is an extra assumption due to the failure of Kodaira vanishing in characteristic p. If X still satisfies Kodaira vanishing (e.g. if X is globally F-split), condition (ii) becomes $\mathrm{H}^i(X,L^m)=0$ for $m,i>0$.

(ii) If X satisfies Kodaira vanishing, (iii) is equivalent to $\mathrm{H}^1(X,B\Omega^d_X\otimes L^m)=0$ for $m>0$, and its Serre dual version is $\mathrm{H}^{d-1}(X,B\Omega^1_X\otimes L^m)=0$ for $m<0$. This follows from exact sequence

$$0\to B\Omega^d_X\to F_*\omega_X\to\omega_X\to 0$$

(iii) Replacing L with L^N for $N \gg 0$ we are in the "sufficiently ample" case where both (ii) and (iii) are trivial by Serre vanishing; in the case of (iii), we have $\mathrm{H}^1(X, B\Omega^d_X \otimes L^{Nm}) = 0$ and hence $\mathrm{H}^0(X, F_*(\omega_X \otimes L^{pNm})) \twoheadrightarrow \mathrm{H}^0(X, \omega_X \otimes L^{Nm})$ for $N \gg 0$ and all $m > 0$, hence $S^0(\omega_X \otimes L^{Nm}) = \mathrm{H}^0(X, \omega_X \otimes L^{Nm})$. Lemma 5.1 then says

$$C(X, L^N) \;\; F\text{-injective} \iff X \;\; \text{Bloch–Kato 1-ordinary}$$

Lemma 5.3. *Suppose $k \geq 1$ and let X be a smooth projective variety, L an ample line bundle. If $\mathrm{H}^i(X, B\Omega^k_X \otimes L^n) = \mathrm{H}^i(X, \Omega^k_X \otimes L^n) = 0$ for all $n \neq 0$ and $1 \leq i \leq i_0$ where i_0 is fixed, then $\mathrm{H}^i(X, B\Omega^{k+1}_X \otimes L^n) = 0$ and $\mathrm{H}^i(X, (F_*\Omega^k_X / B\Omega^k_X) \otimes L^n) = 0$ for all $n \neq 0$ and $1 \leq i \leq i_0 - 1$.*

Proof. This follows at once from exact sequences

$$0 \to B\Omega^k_X \to Z\Omega^k_X \to \Omega^k_X \to 0$$
$$0 \to Z\Omega^k_X \to F_*\Omega^k_X \to B\Omega^{k+1}_X \to 0$$
$$0 \to \Omega^k_X \to F_*\Omega^k_X / B\Omega^k_X \to B\Omega^{k+1}_X \to 0$$

by twisting them by L^n. Explicitly, $\mathrm{H}^i(X, Z\Omega^k_X \otimes L^n) = 0$ for $1 \leq i \leq i_0$ from our assumptions, thus via exact sequence $\mathrm{H}^i(\Omega^k_X \otimes L^{np}) \to \mathrm{H}^i(X, B\Omega^{k+1}_X \otimes L^n) \to \mathrm{H}^{i+1}(X, Z\Omega^k_X \otimes L^n)$ it follows $\mathrm{H}^i(X, B\Omega^{k+1}_X \otimes L^n) = 0$ and $\mathrm{H}^i(X, F_*\Omega^k_X / B\Omega^k_X \otimes L^n) = 0$ for $1 \leq i \leq i_0 - 1$. □

Proposition 5.4. *If $C(X, L)$ is pre-(k, F)-injective, then X is skew $(k+1)$-ordinary.*

Proof. Inducting on k with Lemma 5.1 as the $k = 0$ case, we can assume that X is skew k-ordinary. We will show X is skew $(k+1)$-ordinary, which in this case means $\mathrm{H}^j(X, B\Omega^{k+1}_X) = 0$ for $j \leq d - k - 1$. We first observe that we have a diagram of exact sequences for any $n \in \mathbb{Z}$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^i_X \otimes L^n & \longrightarrow & \frac{F_*\Omega^i_X}{B\Omega^i_X} \otimes L^n & \longrightarrow & B\Omega^{i+1}_X \otimes L^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\pi_*\Omega^i_U)_n & \longrightarrow & \left(\pi_* \frac{F_*\Omega^i_U}{B\Omega^i_U}\right)_{pn} & \longrightarrow & \left(\pi_* B\Omega^{i+1}_U\right)_{pn} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{i-1}_X \otimes L^n & \longrightarrow & \frac{F_*\Omega^{i-1}_X}{B\Omega^{i-1}_X} \otimes L^n & \longrightarrow & B\Omega^i_X \otimes L^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The left column is exact by Proposition 4.5, the middle column is exact by Lemma 4.7 and the right column is exact by Corollary 4.6.1. The exactness of the top row follows from

$$0 \to \frac{Z\Omega^i_X}{B\Omega^i_X} \to \frac{F_*\Omega^i_X}{B\Omega^i_X} \to B\Omega^{i+1}_X \to 0 \tag{34}$$

and from the Cartier isomorphism $C^{-1} : \Omega^i_X \xrightarrow{\simeq} \frac{Z\Omega^i_X}{B\Omega^i_X}$; the same argument in degree $i-1$ yields exactness of the bottom row. For the middle row we argue as follows:

Claim 1: $(\pi_*\Omega^i_U)_n \simeq \left(\pi_* \frac{Z\Omega^i_U}{B\Omega^i_U}\right)_{pn}$. This follows from a calculation on local sections, namely by working over an affine open $V \subset X$, a section $T^n\omega_1 + T^{n-1}\omega_2 \wedge dT \in \Gamma(V, (\pi_*\Omega^i_U)_n)$ is sent by $\pi_* C_U^{-1}$ to a class $[T^{pn}C_X^{-1}(\omega_1) + T^{pn-1}C_X^{-1}(\omega_2) \wedge dT]$ in $\Gamma(V, (\pi_* Z\Omega^i_U/B\Omega^i_U)_{pn})$ via the properties of Cartier maps C_U and C_X. Then exactness of the middle row follows from applying (34) to the weight pn components, since the maps respect grading and π_* is exact.

Claim 2: $(\pi_* F_* \Omega^i_U)_n \simeq F_*(\pi_*\Omega^i_U)_n$. This follows from $\pi_* F_* = F_* \pi_*$; the identification changes the $\mathcal{O}_X$-module structure on the weight component but F_* respects the grading convention (see Remark 4.4).

Now set $n = 0$. The cohomology long exact sequence of the right column in the diagram yields

$$\mathrm{H}^j(X, B\Omega^{k+1}_X) \simeq \mathrm{H}^j(X, (\pi_* B\Omega^{k+1}_U)_0), \quad \mathrm{H}^j(X, (\pi_* B\Omega^k_U)_0) = 0 \quad \text{for } j \leq d-k \tag{35}$$

since $\mathrm{H}^j(X, B\Omega^k_X) = 0$ for $j \leq d-k$ by the inductive hypothesis. By Claim 2 and Corollary 4.6.1 we also have

$$\mathrm{H}^j(X, (\pi_*\Omega^k_U)_0) \simeq \mathrm{H}^j(X, (\pi_* F_* \Omega^k_U)_0) \simeq \mathrm{H}^j\left(X, \left(\pi_* \frac{F_*\Omega^k_U}{B\Omega^k_U}\right)_0\right) \quad \text{for } j+1 \leq d-k \tag{36}$$

where the first isomorphism follows by exactness of F_*.

Now $C(X, L)$ is pre-(k, F)-injective by assumption, so passing to the graded components we have that

$$C_j^{-1} : \mathrm{H}^j(X, (\pi_*\Omega^k_U)_0) \to \mathrm{H}^j\left(X, \left(\pi_* \frac{F_*\Omega^k_U}{B\Omega^k_U}\right)_0\right) \tag{37}$$

is injective for $j \leq d-k$. When $j \leq d-k-1$, the source and target are of equal dimension by (36) so (37) is an isomorphism. The map in (37) fits into the cohomology long exact sequence of the middle row in the diagram, whence

$$\mathrm{H}^j(X, (\pi_* B\Omega^{k+1}_U)_0) = 0, \quad j \leq d-k-1$$

The boundary $j = d-k-1$ is achieved by noting (37) is still injective for $j = d-k$. We conclude from (35) that $\mathrm{H}^j(X, B\Omega^{k+1}_X) = 0$ for $j+k+1 \leq d$, which proves X is skew $(k+1)$-ordinary. □

We next explain a characteristic p phenomenon on boundary maps that will be important in Lemma 5.8 as preparation for Theorem 5.9. Recall we have an exact sequence

$$0 \to \Omega^i_X \otimes L^m \to (\pi_*\Omega^i_U)_0 \otimes L^m \xrightarrow{\mathrm{res}} \Omega^{i-1}_X \otimes L^m \to 0 \tag{38}$$

by Proposition 4.5 and Lemma 4.10, after passing to cohomology we get boundary maps

$$\partial(i, m) : \mathrm{H}^0(X, \Omega^{i-1}_X \otimes L^m) \to \mathrm{H}^1(X, \Omega^i_X \otimes L^m) \tag{39}$$

The boundary maps are $\cup c_1(L)$ followed by the wedge map on differential forms; for example, $\partial(1,0) : \mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^1(X, \Omega^1_X)$ takes $1 \mapsto c_1(L)$.

Lemma 5.5. *Sequence* (38) *splits in* Vect_k *when* $p \nmid m$ *or in* $\mathrm{Mod}_{\mathcal{O}_X}$ *when* $c_1(L) = 0$.

Proof. The natural map

$$\phi : \Omega_X^{i-1} \otimes L^m \hookrightarrow (\pi_*\Omega_U^{i-1})_m \xrightarrow{d_U} (\pi_*\Omega_U^i)_m \simeq (\pi_*\Omega_U^i)_0 \otimes L^m$$

yields $\mathrm{res} \circ \phi = m\mathrm{id}$, so when m is invertible in k this provides a k-linear splitting of (38) and consequently $\partial(i, m) = 0$ for all i; this makes *all* boundary maps vanish in characteristic zero (when $m \neq 0$) as in [SVV23, §A.2].

For the second claim, by construction the extension class of

$$0 \to \Omega_X^1 \to (\pi_*\Omega_U^1)_0 \to \mathcal{O}_X \to 0$$

is identified with $c_1(L) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \Omega_X^1) \simeq \mathrm{H}^1(X, \Omega_X^1)$ so $c_1(L) = 0$ implies this exact sequence splits $\mathcal{O}_X$-linearly. Since $(\pi_*\Omega_U^i)_0 \simeq \wedge^i(\pi_*\Omega_U^1)_0$ we get that (38) splits as well. □

Proposition 5.6. *Suppose $C(X, L)$ is pre-(k, F)-injective, then X satisfies*

(i) $\mathrm{H}^i(X, (\pi_*\Omega_U^j)_0 \otimes L^m) = 0$ *for* $m \neq 0$, $j \leq k$ *and* $1 \leq i < d - j$
(ii) $\mathrm{H}^i(X, (\pi_* B\Omega_U^{j+1})_0 \otimes L^m) = 0$ *for* $m \neq 0$, $j \leq k$ *and* $1 \leq i < d - j$

Proof. Inducting on k with base case $k = 0$ (Lemma 5.1), we once again reduce to verifying (i)-(ii) for $j = k$, assuming they hold for $j \leq k - 1$.

First, note $\mathrm{H}^i(X, (\pi_*\Omega_U^k)_m) \hookrightarrow \mathrm{H}^i\left(X, \left(\pi_* F_*\Omega_U^k / B\Omega_U^k\right)_{pm}\right)$ *for* $i \leq d - k$. This follows from pre-(k, F)-injectivity of $C(X, L)$ as $\alpha : \mathrm{H}^i(U, \Omega_U^k) \hookrightarrow \mathrm{H}^i(U, F_*\Omega_U^k / B\Omega_U^k)$ is injective for $i \leq d - k$. Note

$$\mathrm{H}^i(U, \Omega_U^k) \simeq \mathrm{H}^i(X, \pi_*\Omega_U^k) \simeq \bigoplus_{m \in \mathbb{Z}} \mathrm{H}^i(X, (\pi_*\Omega_U^k)_m)$$

$$\mathrm{H}^i\left(U, \frac{F_*\Omega_U^k}{B\Omega_U^k}\right) \simeq \mathrm{H}^i\left(X, \pi_*\frac{F_*\Omega_U^k}{B\Omega_U^k}\right) \simeq \bigoplus_{m \in \mathbb{Z}} \mathrm{H}^i\left(X, \left(\pi_*\frac{F_*\Omega_U^k}{B\Omega_U^k}\right)_m\right)$$

via the $\mathbb{Z}$-gradings obtained in Section 4 (see Proposition 4.5 and Lemma 4.7), and map α splits into maps

$$\alpha_m : \mathrm{H}^i(X, (\pi_*\Omega_U^k)_m) \hookrightarrow \mathrm{H}^i\left(X, \left(\pi_*\frac{F_*\Omega_U^k}{B\Omega_U^k}\right)_{pm}\right) \tag{40}$$

which are then also injective for $i \leq d - k$.

Now fix $m \neq 0$, $1 \leq i < d - k$ and note

$$\mathrm{H}^i(X, (\pi_*\Omega_U^k)_m) \hookrightarrow \mathrm{H}^i(X, (\pi_*\Omega_U^k)_{pm}) = \mathrm{H}^i(X, (\pi_*\Omega_U^k)_0 \otimes L^{pm})$$

in this region: via Corollary 4.6.1, taking weight $n = pm$ and passing to cohomology yields

$$\cdots \mathrm{H}^i(X, (\pi_* B\Omega_U^k)_0 \otimes L^m) \to \mathrm{H}^i(X, F_*((\pi_*\Omega_U^k)_0 \otimes L^{pm})) \xrightarrow{\alpha} \mathrm{H}^i\left(X, \left(\pi_*\frac{F_*\Omega_U^k}{B\Omega_U^k}\right)_0 \otimes L^m\right) \cdots$$

and by the inductive assumption $\mathrm{H}^i(X, (\pi_* B\Omega_U^k)_0 \otimes L^m) = 0$ for $1 \leq i < d - k + 1$ so α is an isomorphism for $1 \leq i < d - k$.

For $m > 0$ we may thus iterate and get $\mathrm{H}^i(X, (\pi_*\Omega_U^k)_0 \otimes L^m) \hookrightarrow \mathrm{H}^i(X, (\pi_*\Omega_U^k)_0 \otimes L^{mp^N})$, and the right hand side vanishes for $N \gg 0$ by Serre vanishing. For $m < 0$ we dually get

$$\mathrm{H}^{d-i}(X, \omega_X \otimes (\pi_*\Omega_U^k)_0^\vee \otimes L^{-mp^N}) \twoheadrightarrow \mathrm{H}^{d-i}(X, \omega_X \otimes (\pi_*\Omega_U^k)_0^\vee \otimes L^{-m})$$

for $i < d$, and for $N \gg 0$ the left hand side vanishes by Serre vanishing again. Altogether we obtain (i) for $j = k$. We next observe

$$0 \to (\pi_*\Omega_U^k)_0 \otimes L^m \overset{C_U^{-1}}{\to} \left(\pi_* \left(\frac{F_*\Omega_U^k}{B\Omega_U^k}\right)\right)_0 \otimes L^m \to (\pi_* B\Omega_U^{k+1})_0 \otimes L^m \to 0 \tag{41}$$

via the Cartier sequence on U and taking weights.

Passing to cohomology we have that $\mathrm{H}^i(C_U^{-1})$ is injective for $i \leq d-k$ by the pre-(k,F)-injectivity condition on $C(X,L)$, and moreover $\mathrm{H}^i(C_U^{-1})$ is identically the zero map for $1 \leq i < d-k$ by (i) since we established

$$\mathrm{H}^i(X, (\pi_*\Omega_U^k)_0 \otimes L^m) \simeq \mathrm{H}^i\left(X, \left(\pi_* \left(\frac{F_*\Omega_U^k}{B\Omega_U^k}\right)\right)_0 \otimes L^m\right) \simeq 0, \quad 1 \leq i < d-k$$

Together with injectivity at $i = d-k$ we obtain $\mathrm{H}^i(X, (\pi_* B\Omega_U^{k+1})_0 \otimes L^m) = 0$ for $1 \leq i < d-k$, i.e. we recover (ii) as well. □

Proposition 5.7. *Assume $d \geq 2$ and $k \leq d-2$. Then $C(X,L)$ is pre-(k,F)-injective if X satisfies the following:*

(i) *X is skew $(k+1)$-ordinary*
(ii) *$\mathrm{H}^i(X, (\pi_*\Omega_U^j)_0 \otimes L^m) = 0$ for $m \neq 0$, $j \leq k$ and $1 \leq i < d-j$*
(iii) *$\mathrm{H}^i(X, (\pi_* B\Omega_U^{j+1})_0 \otimes L^m) = 0$ for $m \neq 0$, $j \leq k$ and $1 \leq i < d-j$.*

Proof. Inducting on k and using Lemma 5.1 as base case $k = 0$, we reduce to checking that

$$\mathrm{H}^i(C^{-1}) : \mathrm{H}^i(U, \Omega_U^k) \to \mathrm{H}^i\left(U, F_*\Omega_U^k / B\Omega_U^k\right) \tag{42}$$

is injective for $1 \leq i \leq d-k$, and that $C : Z\Omega^{[i]}_{C(X,L)} \to \Omega^{[i]}_{C(X,L)}$ is surjective for $i \leq k$. As in the proof of Proposition 5.6, passing to the weight decomposition it suffices to show

$$\alpha_m^i : \mathrm{H}^i(X, (\pi_*\Omega_U^k)_m) \hookrightarrow \mathrm{H}^i\left(X, \left(\pi_* \frac{F_*\Omega_U^k}{B\Omega_U^k}\right)_{pm}\right), \quad i \leq d-k \tag{43}$$

is injective for all m. By the short exact sequence in (41), we see that

$$\ker \alpha_m^i = \mathrm{im}(\mathrm{H}^{i-1}(X, (\pi_* B\Omega_U^{k+1})_0 \otimes L^m) \to \mathrm{H}^i(X, (\pi_*\Omega_U^k)_m))$$

and for $2 \leq i \leq d-k$ this vanishes: when $m \neq 0$ this is condition (iii), and for $m = 0$ this follows from exactness of

$$\cdots \mathrm{H}^{i-1}(X, B\Omega_X^{k+1}) \to \mathrm{H}^{i-1}(X, (\pi_* B\Omega_U^{k+1})_0) \to \mathrm{H}^{i-1}(X, B\Omega_X^k) \cdots$$

and skew $(k+1)$-ordinarity, i.e. condition (i). For $i = 1$, when $m \neq 0$ the source of α_m^i is zero by condition (ii) (cf. Lemma 4.10), while for $m = 0$ we have $\mathrm{H}^0(X, (\pi_* B\Omega_U^{k+1})_0) = 0$ via the exact sequence

$$0 \to B\Omega_X^{k+1} \to (\pi_* B\Omega_U^{k+1})_0 \to B\Omega_X^k$$

and skew $(k+1)$-ordinarity i.e. condition (i). Thus (43) is injective for $1 \leq i \leq d-k$ and all m. Finally, surjectivity of C follows from (iii) and Corollary 4.11.1 when $m \neq 0$, and from skew $(k+1)$-ordinarity when $m = 0$, so we conclude $C(X,L)$ is pre-(k,F)-injective. □

From Propositions 5.4, 5.6 and 5.7 we have:

Corollary 5.7.1. *For $d \geq 2$ and $k \leq d-2$, $C(X,L)$ is pre-(k,F)-injective if and only if*

(i) *X is skew $(k+1)$-ordinary*

(ii) $\mathrm{H}^i(X, (\pi_*\Omega_U^j)_0 \otimes L^m) = 0$ *for* $m \neq 0$, $j \leq k$ *and* $1 \leq i < d - j$
(iii) $\mathrm{H}^i(X, (\pi_* B\Omega_U^{j+1})_0 \otimes L^m) = 0$ *for* $m \neq 0$, $j \leq k$ *and* $1 \leq i < d - j$.

If $\Omega^i_{C(X,L)}$ *is reflexive for* $i \leq k$, *then* $C(X, L)$ *is* (k, F)*-injective if and only if* (i)-(iii) *above hold.*

We now explain how we can obtain a stronger cohomological vanishing result in terms of sheaves intrinsically defined on X. The main point is that the boundary maps $\partial(j, m)$ defined in (39) should vanish.

Lemma 5.8. *Assume* $k \leq d-2$ *and suppose boundary maps* $\partial(j, m)$ *are zero for* $j \leq k, m \neq 0$, *then the following vanishings are equivalent:*

(i) $\mathrm{H}^i(X, (\pi_*\Omega_U^j)_0 \otimes L^m) = \mathrm{H}^i(X, (\pi_* B\Omega_U^{j+1})_0 \otimes L^m) = 0$ *for* $m \neq 0$, $j \leq k$, $1 \leq i < d - j$
(ii) $\mathrm{H}^i(X, \Omega_X^j \otimes L^m) = \mathrm{H}^i(X, B\Omega_X^{j+1} \otimes L^m) = 0$ *for* $m \neq 0$, $j \leq k$, $1 \leq i < d - j$

Proof. It is clear that (ii) always implies (i) by looking at the associated exact sequences

$$0 \to \Omega_X^j \otimes L^m \to (\pi_*\Omega_U^j)_0 \otimes L^m \xrightarrow{\mathrm{res}} \Omega_X^{j-1} \otimes L^m \to 0$$

$$0 \to B\Omega_X^{j+1} \otimes L^m \to (\pi_* B\Omega_U^{j+1})_0 \otimes L^m \xrightarrow{\mathrm{res}} B\Omega_X^j \otimes L^m \to 0$$

For the converse, assume $\partial(j, m) = 0$ as above and (i). We then have

$$H^1(X, \Omega_X^j \otimes L^m) \hookrightarrow \mathrm{H}^1(X, (\pi_*\Omega_U^j)_0 \otimes L^m) = 0$$

by assumption, and as $H^i(X, (\pi_*\Omega_U^j)_0 \otimes L^m) = 0$ for i, j, m as above, we get

$$\mathrm{H}^{i-1}(X, \Omega_X^{j-1} \otimes L^m) \simeq \mathrm{H}^i(X, \Omega_X^j \otimes L^m), \quad 2 \leq i < d - j$$

where the left hand side is inductively zero, so we get the first half of (ii). For the other half, we show inductively on j that $\mathrm{H}^i(X, B\Omega_X^{j+1} \otimes L^m) = 0$ for $m \neq 0, j \leq k, 1 \leq i < d - j$.

The case $j = 0$ is obvious as $B\Omega_X^1 \simeq (\pi_* B\Omega_U^1)_0$. As before, by assumption we have

$$\mathrm{H}^{i-1}(X, B\Omega_X^j \otimes L^m) \simeq \mathrm{H}^i(X, B\Omega_X^{j+1} \otimes L^m), \;\; 2 \leq i < d - j$$

and the left hand side vanishes inductively, so what is left is to verify $\mathrm{H}^1(X, B\Omega_X^{j+1} \otimes L^m) = 0$.

Note that by induction, the Cartier sequence

$$0 \to B\Omega_X^{j-1} \otimes L^m \to Z\Omega_X^{j-1} \otimes L^m \to \Omega_X^{j-1} \otimes L^m \to 0$$

yields $\mathrm{H}^1(X, Z\Omega_X^{j-1} \otimes L^m) = 0$. Therefore by exact sequence

$$0 \to Z\Omega_X^{j-1} \otimes L^m \to F_*\Omega_X^{j-1} \otimes L^m \to B\Omega_X^j \otimes L^m \to 0$$

we obtain a surjection $\mathrm{H}^0(X, F_*(\Omega_X^{j-1} \otimes L^{mp})) \twoheadrightarrow \mathrm{H}^0(X, B\Omega_X^j \otimes L^m)$. Hence a section ω in the right hand side lifts to a section $F_*\eta \in \mathrm{H}^0(X, F_*(\Omega_X^{j-1} \otimes L^{mp}))$; since $\partial(j, pm) = 0$, the underlying section $\eta \in \Gamma(\Omega_X^{j-1} \otimes L^{mp})$ lifts to a section $\delta \in \mathrm{H}^0(X, (\pi_*\Omega_U^j)_0 \otimes L^{mp})$. We then compute $\mathrm{res}(d\delta) = d(F_*\eta) = \omega$, so we have a surjection

$$\mathrm{H}^0(X, (\pi_* B\Omega_U^{j+1})_0 \otimes L^m) \twoheadrightarrow \mathrm{H}^0(X, B\Omega_X^j \otimes L^m)$$

and so $\mathrm{H}^1(X, B\Omega_X^{j+1} \otimes L^m) \hookrightarrow \mathrm{H}^1(X, (\pi_* B\Omega_U^{j+1})_0 \otimes L^m) = 0$, yielding (ii) in full. □

Now observe that given L, the cone $C(X, L^p)$ satisfies $c_1(L^p) = pc_1(L) = 0$, thus by Lemma 5.5 the boundary maps $\partial(i, m) = 0$. In view of Corollary 5.7.1 and Lemma 5.8 we deduce the following statement:

Theorem 5.9. *Let X be a smooth projective variety of dimension $d \geq 2$ and $L \in \mathrm{Pic}(X)$ ample. Consider the affine Frobenius-twisted cone $C(X, L^p)$. Then for $k \leq d-2$ we have that $C(X, L^p)$ is pre-(k, F)-injective if and only if*

(i) *X is skew $(k+1)$-ordinary*
(ii) *$\mathrm{H}^i(X, \Omega_X^j \otimes L^{mp}) = \mathrm{H}^i(X, B\Omega_X^{j+1} \otimes L^{mp}) = 0$ for $m \neq 0$, $j \leq k$ and $1 \leq i < d-j$.*

More generally, assume $c_1(L) = 0$. Then for $k \leq d-2$, $C(X, L)$ is pre-(k, F)-injective if and only if (i) *and* (ii) *hold with L^{mp} replaced by L^m.*

Remark 5.10. Assume the more general setting of the cone $C(X, L)$ and $c_1(L) = 0$. By Lemma 5.3, we can relax the $B\Omega^\bullet$ vanishing condition in (ii) to

$$\mathrm{H}^{d-j-1}(X, B\Omega_X^{j+1} \otimes L^m) = 0$$

for $j \leq k$, or its dual version $\mathrm{H}^{j+1}(X, B\Omega_X^{d-j} \otimes L^m) = 0$ for $j \leq k$. This implies

$$C : \mathrm{H}^j(X, Z\Omega_X^{d-j} \otimes L^m) \twoheadrightarrow \mathrm{H}^j(X, \Omega_X^{d-j} \otimes L^m), \quad j \leq k$$

Note for $k = 0$ this recovers Lemma 5.1 (iii). We may rephrase this vanishing in terms of 'higher' S^0 conditions: recall the higher Cartier operator $C_n : Z_n\Omega_X^j \to \Omega_X^j$ from Section 2.1. Define

$$S^j(\Omega_X^{d-j} \otimes L^m) := \bigcap_{n \geq 1} \mathrm{im}\left(\mathrm{H}^j(X, Z_n\Omega_X^{d-j} \otimes L^m) \xrightarrow{C_n} \mathrm{H}^j(X, \Omega_X^{d-j} \otimes L^m)\right)$$

For $j = 0$ this is the usual S^0 as in (31). Then in (ii) we may replace $\mathrm{H}^i(X, B\Omega_X^{j+1} \otimes L^m) = 0$ with the condition

$$S^j(\Omega_X^{d-j} \otimes L^m) = \mathrm{H}^j(X, \Omega_X^{d-j} \otimes L^m), \qquad m \neq 0,\ 0 \leq j \leq k$$

i.e. the "higher Cartier-stable" cohomology classes span all of $\mathrm{H}^j(X, \Omega_X^{d-j} \otimes L^m)$.

We note the pre-(k, F)-injectivity of $C(X, L^p)$ in Theorem 5.9 may not be upgraded to (k, F)-injectivity for $k \geq 1$ as $\Omega^1_{C(X,L^p)}$ is *not* reflexive. For the sake of completion we mention separately the case of cones over smooth curves (i.e. $d = 1$):

Proposition 5.11. *Suppose X is a smooth curve, $L \in \mathrm{Pic}(X)$ and $C(X, L)$ the affine cone over X with conormal L.*

(i) *$C(X, L)$ is pre-(k, F)-injective for some (equivalently all) $k \geq 1$ if and only if $C(X, L)$ is F-injective.*
(ii) *$C(X, L)$ is (k, F)-injective for some (equivalently all) $k \geq 1$ if and only if it is smooth.*

Proof. In order to have surjectivity of the reflexive Cartier map $C : Z\Omega^{[i]}_{C(X,L)} \to \Omega^{[i]}_{C(X,L)}$, by affineness of $C(X, L)$ we reduce (as in the proof of Lemma 4.11) to showing that $\Gamma(\pi_* Z\Omega_U^i) \to \Gamma(\pi_*\Omega_U^i)$ is surjective.

For (i), only F-injective $\Rightarrow$ pre-(k, F)-injective is nontrivial. By Lemma 5.1 we have $\mathrm{H}^1(X, L^m) = 0$ for $m > 0$ and $\mathrm{H}^1(X, B\Omega_X^1) = 0$. Via exact sequence

$$0 \to L^m \to F_*\mathcal{O}_X \otimes L^m \to B\Omega_X^1 \otimes L^m \to 0$$

we therefore get $\mathrm{H}^1(X, B\Omega_X^1 \otimes L^m) = 0$ for $m \geq 0$. Moreover as $\Gamma((\pi_*\Omega_U^1)_m) \simeq \Gamma(\omega_X \otimes L^m)$ for $m < 0$, we have

$$\Gamma((\pi_*\Omega_U^1)_m) \simeq \Gamma(\omega_X \otimes L^m) \simeq \mathrm{H}^1(X, L^{-m})^\vee = 0, \quad m < 0$$

so altogether by Lemma 4.11 we get that $\Gamma(\pi_* Z\Omega^1_U) \to \Gamma(\pi_*\Omega^1_U)$ is surjective. Similarly in degree 2 we have

$$\Gamma\left(\bigoplus_m \left(\pi_* Z\Omega^2_U\right)_m\right) \to \Gamma\left(\bigoplus_m \left(\pi_*\Omega^2_U\right)_m\right) \simeq \Gamma\left(\bigoplus_m (\omega_X \otimes L^m)\right)$$

so for surjectivity of the reflexive Cartier map and $(\pi_* Z\Omega^2_U)_{pm} \simeq F_*\omega_X \otimes L^m$ it suffices to have $\Gamma(F_*\omega_X \otimes L^m) \to \Gamma(\omega_X \otimes L^m)$ surjective. For $m > 0$ this is Lemma 5.1 (iii), for $m = 0$ it follows from 1-ordinarity and for $m < 0$ the target is Serre dual to $\mathrm{H}^1(X, L^{-m})^\vee = 0$. Thus the reflexive Cartier map on $C(X, L)$ is always surjective.

Now pre-(k, F)-injectivity translates to $\mathrm{H}^j_{\mathfrak{m}}(\Omega^{[i]}_{C(X,L)}) \to \mathrm{H}^j_{\mathfrak{m}}\left(F_*\Omega^{[i]}_{C(X,L)}/B\Omega^{[i]}_{C(X,L)}\right)$ injective for $i \le k$, $j \le 2 - i$, so when $i > 0$ this is trivially true by Lemma 4.3 and for $i = 0$ it is the definition of F-injectivity. This completes the proof of (i).

For (ii), by definition if $C(X, L)$ is (k, F)-injective then $\Omega^1_{C(X,L)}$ is reflexive. Denote the section ring by $R = \bigoplus_{m\ge 0} \mathrm{H}^0(X, L^m)$ so $(\Omega^1_{R/k})_1 = d(R_1)$ i.e. it is generated by elements of the form $df, f \in \mathrm{H}^0(X, L)$. By reflexivity, $(\Omega^1_{R/k})_1 \simeq \mathrm{H}^0(X, (\pi_*\Omega^1_U)_1)$ and by looking at the exact sequence of Proposition 4.5

$$0 \to \omega_X \otimes L \to (\pi_*\Omega^1_U)_1 \xrightarrow{\mathrm{res}} L \to 0$$

we have on global sections that $\mathrm{res}(df) = f$ i.e. the residue map is injective. Thus by Riemann–Roch,

$$0 = h^0(X, \omega_X \otimes L) = g(X) - 1 + \deg(L)$$

forces $g(X) = 0$ and $\deg(L) = 1$. We conclude $C(X, L) = C(\mathbb{P}^1, \mathcal{O}(1))$ is smooth. □

Corollary 5.11.1. *Suppose $L \in \mathrm{Pic}(X)$ is sufficiently ample in the sense that $L = L_0^N$ for L_0 ample and $N \gg 0$. Then $C(X, L)$ is pre-(k, F)-injective if and only if X is skew $(k + 1)$-ordinary.*

Proof. By Proposition 5.4 we have the $\Longrightarrow$ direction. For the converse, assume X is skew $(k+1)$-ordinary, then we need to verify the Cartier surjectivity and inverse Cartier injectivity conditions on $C(X, L)$ (see Definition 2.10). We have an exact sequence

$$0 \to (\pi_* B\Omega^j_U)_0 \otimes L^m \to (\pi_* Z\Omega^j_U)_0 \otimes L^m \xrightarrow{C} (\pi_*\Omega^j_U)_0 \otimes L^m \to 0$$

for each weight $m \in \mathbb{Z}$. Taking cohomology we note that the induced map $\mathrm{H}^0(C)$ is surjective for $j \le k$ when $m = 0$ by $\mathrm{H}^1(X, (\pi_* B\Omega^j_U)_0) = 0$; this follows from skew $(k + 1)$-ordinarity of X via the residue exact sequence

$$0 \to B\Omega^j_X \to (\pi_* B\Omega^j_U)_0 \to B\Omega^{j-1}_X \to 0 \tag{44}$$

noting also that $\mathrm{H}^1(X, B\Omega^d_X) \simeq \mathrm{H}^{d-1}(X, B\Omega^1_X)^\vee = 0$. $H^0(C)$ is also surjective when $m > 0$ as $\mathrm{H}^1(X, (\pi_* B\Omega^j_U)_0 \otimes L^m) = 0$ by sufficient ampleness of L. Sufficient ampleness also yields $\mathrm{H}^1(X, (\pi_* B\Omega^j_U)_0 \otimes L^m) = 0$ when $m < 0$ and $d \ge 2$ so by surjectivity on each weight we conclude that the Cartier map $C : Z\Omega^{[j]}_{C(X,L)} \to \Omega^{[j]}_{C(X,L)}$ is surjective for $j \le k$ by the proof of Lemma 4.11 (cf. Corollary 4.11.1). When $d = 1$ and $m < 0$ we note instead that $\mathrm{H}^0(X, (\pi_*\Omega^j_U)_0 \otimes L^m)^\vee \simeq \mathrm{H}^d(X, \omega_X \otimes (\pi_*\Omega^j_U)_0^\vee \otimes L^{-m}) = 0$ by sufficient ampleness of L, so $\mathrm{H}^0(C)$ is still surjective in all weights m.

For the inverse Cartier injectivity conditions, consider sequence

$$0 \to (\pi_*\Omega_U^j)_0 \otimes L^m \xrightarrow{C^{-1}} \left(\pi_*\left(\frac{F_*\Omega_U^j}{B\Omega_U^j}\right)\right)_0 \otimes L^m \to (\pi_* B\Omega_U^{j+1})_0 \otimes L^m \to 0$$

Taking cohomology again, we see that for $j \leq k$ and $1 \leq i \leq d-j$ we have the following:

- If $m > 0$, $\mathrm{H}^i(X, (\pi_*\Omega_U^j)_0 \otimes L^m) = 0$ by sufficient ampleness of L so $\mathrm{H}^i(C^{-1})$ is trivially injective.
- If $m = 0$ we note $\mathrm{H}^{i-1}(X, (\pi_* B\Omega_U^{j+1})_0) = 0$ by skew $(k+1)$-ordinarity of X and residue sequence (44) so $\mathrm{H}^i(C^{-1})$ is injective.
- If $m < 0$ then $\mathrm{H}^{i-1}(X, (\pi_* B\Omega_U^{j+1})_0 \otimes L^m) = 0$ since its dual

  $$\mathrm{H}^{i-1}(X, (\pi_* B\Omega_U^{j+1})_0 \otimes L^m)^\vee \simeq \mathrm{H}^{d-i+1}(X, \omega_X \otimes (\pi_* B\Omega_U^{j+1})_0^\vee \otimes L^{-m}) = 0$$

  by sufficient ampleness of L (and $d \geq i$), so $\mathrm{H}^i(C^{-1})$ is injective.

We conclude that $\mathrm{H}^i(C^{-1})$ is injective for all weights m i.e.

$$\mathrm{H}^i_{\mathfrak{m}}(\Omega^{[j]}_{C(X,L)}) \to \mathrm{H}^i_{\mathfrak{m}}(F_*\Omega^{[j]}_{C(X,L)}/B\Omega^{[j]}_{C(X,L)})$$

for $j \leq k$ and $2 \leq i \leq d+1-j$. By reflexivity, the local cohomology in the source vanishes for $i \leq 1$ so we conclude that $C(X, L)$ is pre-(k, F)-injective. □

6. (k, F)-rationality for cones

As in the previous section, we are in the setting of Definition 4.1 and we let X be a smooth projective d-dimensional variety over a perfect field k of characteristic p, L an ample line bundle on X and $C(X, L)$ the affine cone over X with conormal bundle L. The goal of this section is to derive a criterion of (k, F)-rationality for cones $C(X, L)$ in terms of X, building on the (k, F)-injectivity criterion from Section 5. The main conclusion is in Proposition 6.5, with Corollary 6.5.1 providing a cleaner criterion under certain assumptions on X.

We make a preliminary observation on the usual F-rationality of cones:

Lemma 6.1. *$C(X, L)$ is F-rational if and only if $C(X, L)$ is F-injective and $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$.*

Proof. This is a restatement of [TW18, Prop. 3.15] once we clarify the grading: by Lemma 4.3 the degree m part of $\mathrm{H}^{i+1}_{\mathfrak{m}}(\mathcal{O}_{C(X,L)})$ is $\mathrm{H}^i(X, (\pi_*\mathcal{O}_U)_m) = \mathrm{H}^i(X, L^m)$ for $i \geq 1$, so $C(X, L)$ is Cohen–Macaulay if and only if $\mathrm{H}^i(X, L^m) = 0$ for all m and $1 \leq i < d$, and in particular the degree zero part of $\mathrm{H}^{d+1}_{\mathfrak{m}}(\mathcal{O}_{C(X,L)})$ is $\mathrm{H}^d(X, \mathcal{O}_X)$. If $C(X, L)$ is F-injective then by Lemma 5.1 we have $\mathrm{H}^i(X, L^m) = 0$ for $m \neq 0$ and $1 \leq i < d$, and so together with $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ we get that $C(X, L)$ is Cohen–Macaulay. The rest follows from [TW18, Prop. 3.15]. □

Lemma 6.2. *Assume $C(X, L)$ is pre-(k, F)-injective, then $\mathrm{H}^j(U, B_n\Omega_U^i) = 0$ for all $n \geq 1$, $i \leq k+1$ and $1 \leq j \leq d-i$.*

Proof. The case $i = 0$ is trivial so we henceforth assume $i \geq 1$. First set $n = 1$ so $B_n\Omega_U^i = B\Omega_U^i$. It suffices to have $\mathrm{H}^j(X, (\pi_* B\Omega_U^i)_m) = 0$ for all weights m. When $p \mid m$, $(\pi_* B\Omega_U^i)_m \simeq (\pi_* B\Omega_U^i)_0 \otimes L^{m/p}$ by Lemma 4.10 and the formulas after its proof, while if $p \nmid m$ we have $(\pi_* B\Omega_U^i)_m \simeq F_*(\Omega_X^{i-1} \otimes L^m)$ by Lemma 4.6. Hence for $m \neq 0$ divisible by p, we have

$\mathrm{H}^j(X, (\pi_* B\Omega^i_U)_m) = 0$ for $i \leq k+1, 1 \leq j \leq d-i$ by Proposition 5.6 (ii), while for $m = 0$ we have exact sequence

$$\cdots \mathrm{H}^j(X, B\Omega^i_X) \to \mathrm{H}^j(X, (\pi_* B\Omega^i_U)_0) \to \mathrm{H}^j(X, B\Omega^{i-1}_X) \cdots$$

and so $\mathrm{H}^j(X, (\pi_* B\Omega^i_U)_0) = 0$ for $i \leq k+1, 1 \leq j \leq d-i$ by skew $(k+1)$-ordinarity of X. When $p \nmid m$, we have that sequence

$$0 \to \Omega^{i-1}_X \otimes L^m \to (\pi_* \Omega^{i-1}_U)_m \to \Omega^{i-2}_X \otimes L^m \to 0$$

splits k-linearly by Lemma 5.5 and so $\mathrm{H}^j(X, \Omega^{i-1}_X \otimes L^m)$ is a summand of $\mathrm{H}^j(X, (\pi_* \Omega^{i-1}_U)_m)$ which vanishes for $i \leq k+1, 1 \leq j \leq d-i$ by Proposition 5.6 and so

$$\mathrm{H}^j(X, (\pi_* B\Omega^i_U)_m) = \mathrm{H}^j(X, F_*(\Omega^{i-1}_X \otimes L^m)) = 0$$

in the same region. Finally, $\mathrm{H}^j(U, B_n\Omega^i_U) = 0$ for $n \geq 2$ and $i \leq k+1, 1 \leq j \leq d-i$ follows inductively from Lemma 2.5 (iii). □

Lemma 6.3. *If $C(X, L)$ is pre-(k, F)-injective, then the iterated inverse Cartier map on U,*

$$C_n^{-1} : \mathrm{H}^j(U, \Omega^i_U) \to \mathrm{H}^j\left(U, F^n_* \Omega^i_U / B_n \Omega^i_U\right) \tag{45}$$

is injective for all $n \geq 1$, $i \leq k$ and $1 \leq j \leq d-i$.

Proof. The argument follows [KW24, Lem. 3.28]. By Lemma 2.5 (iii) we have for $n \geq 2$ that $B_{n-1}\Omega^i_U \xrightarrow{\sim} B_n\Omega^i_U / F^{n-1}_* B\Omega^i_U$ and this isomorphism fits into a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_{n-1}\Omega^i_U & \longrightarrow & F^{n-1}_*\Omega^i_U & \longrightarrow & F^{n-1}_*\Omega^i_U / B_{n-1}\Omega^i_U & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle F^{n-1}_* C_U^{-1}} & & \downarrow{\scriptstyle (*)} & & \\
0 & \longrightarrow & B_n\Omega^i_U / F^{n-1}_* B\Omega^i_U & \longrightarrow & F^{n-1}_*\left(F_*\Omega^i_U / B\Omega^i_U\right) & \longrightarrow & F^n_*\Omega^i_U / B_n\Omega^i_U & \longrightarrow & 0
\end{array}$$

The middle map $F^{n-1}_* C_U^{-1}$ induces injections on cohomology, by pre-(k, F)-injectivity. Thus by the four lemma $(*)$ induces injective maps $\mathrm{H}^j(U, F^{n-1}_*\Omega^i_U / B_{n-1}\Omega^i_U) \to \mathrm{H}^j(U, F^n_*\Omega^i_U / B_n\Omega^i_U)$ for $i \leq k$ and $1 \leq j \leq d-i$. Iterating these injections for all n and precomposing with C_U^{-1}, which is injective by pre-(k, F)-injectivity, we see that (45) is injective for such i, j. □

Lemma 6.4. *If $C(X, L)$ is pre-(k, F)-injective for $k \leq d-2$, then there is an $N \geq 1$ such that for all $n \geq N$ we have $\mathrm{H}^j(X, (\pi_* Z_n\Omega^i_U)_m) = 0$ for $m \neq 0$, $1 \leq i \leq k$ and $1 \leq j \leq d-i$.*

Proof. First, note that by Lemma 4.9, iterating the Cartier isomorphism $\Omega^i_U \simeq Z_n\Omega^i_U / B_n\Omega^i_U$ yields a weight identification

$$\left(\pi_* \left(\frac{Z_n\Omega^i_U}{B_n\Omega^i_U}\right)\right)_m \simeq \begin{cases} (\pi_*\Omega^i_U)_{m/p^n}, & p^n \mid m, \\ 0, & p^n \nmid m. \end{cases}$$

Thus if $p^n \nmid m$, we have $\mathrm{H}^j(X, (\pi_* Z_n\Omega^i_U)_m) = \mathrm{H}^j(X, (\pi_* B_n\Omega^i_U)_m) = 0$ for $n \geq 1$, $i \leq k$ and $1 \leq j \leq d-i$ by Lemma 6.2. So the only weight cases that remain are those divisible by p^n.

By Lemmas 6.2 and 6.3 we have that the composite map

$$\mathrm{H}^j(U, Z_n\Omega^i_U) \to \mathrm{H}^j(U, \Omega^i_U) \to \mathrm{H}^j(U, F^n_*\Omega^i_U / B_n\Omega^i_U)$$

is injective for $i \leq k$ and $1 \leq j \leq d-i$, and this composite map factors through

$$\mathrm{H}^j(U, Z_n\Omega^i_U) \xrightarrow{(*)} \mathrm{H}^j(U, F^n_*\Omega^i_U) \to \mathrm{H}^j(U, F^n_*\Omega^i_U / B_n\Omega^i_U)$$

so $(*)$ is injective in the same region. Passing to the respective weight decompositions of each sheaf, we have an injection $\mathrm{H}^j(X,(\pi_*Z_n\Omega^i_U)_m)\to \mathrm{H}^j(X,F^n_*((\pi_*\Omega^i_U)_0\otimes L^m))$ for all m and i,j,n as above (cf. Lemma 4.10). By Serre vanishing (and Serre duality when $m<0$), we can find $N\gg 1$ so that for $|m|\geq p^N$ we have $\mathrm{H}^j(X,(\pi_*\Omega^i_U)_0\otimes L^m)=0$ for all $1\leq i\leq k$ and $1\leq j\leq d-i$; note that $(\pi_*\Omega^i_U)_0$ is locally free. Then for $n\geq N$ with $p^n\,|\,m$ we have $|m|\geq p^N$ so $\mathrm{H}^j(X,(\pi_*Z_n\Omega^i_U)_m)\hookrightarrow \mathrm{H}^j(X,F^n_*((\pi_*\Omega^i_U)_0\otimes L^m))=0$. Thus we obtain the vanishing of $\mathrm{H}^j(X,(\pi_*Z_n\Omega^i_U)_m)$ for $n\geq N$, all $m\neq 0$ and i,j as above. □

Proposition 6.5. *Let X be a smooth projective variety and $L\in\mathrm{Pic}(X)$ ample. For $k<\frac{d}{2}$, the cone $C(X,L)$ is (k,F)-rational if and only if*

(i) *X is skew $(k+1)$-ordinary*
(ii) *$\mathrm{H}^i(X,(\pi_*\Omega^j_U)_0\otimes L^m)=\mathrm{H}^i(X,(\pi_*B\Omega^{j+1}_U)_0\otimes L^m)=0$ for $m\neq 0$, $j\leq k$, $1\leq i<d-j$*
(iii) *$\mathrm{H}^j(X,\mathcal{O}_X)=0$ for $j>0$ and there exists an $N\geq 1$ such that $\mathrm{H}^j(X,(\pi_*Z_n\Omega^i_U)_0)=0$ for $n\geq N$, $1\leq i\leq k$ and $1\leq j\leq d-i$.*

Proof. Assume first that $d=1$ (so $k=0$). Conditions (i)-(iii) boil down to $\mathrm{H}^1(X,\mathcal{O}_X)=0$ as genus zero curves are skew 1-ordinary, so X is a conic that becomes isomorphic to $\mathbb{P}^1$ after a finite extension K. Write $X_K\simeq\mathbb{P}^1_K$ and $L_K\simeq\mathcal{O}_{\mathbb{P}^1}(N)$ (with $N>0$ by ampleness). We can compute

$$\bigoplus_{m\geq 0}\mathrm{H}^0(X,L^m)\otimes K\simeq\bigoplus_{m\geq 0}\mathrm{H}^0(\mathbb{P}^1_K,\mathcal{O}(Nm))\simeq K[x^N,x^{N-1}y,\cdots,y^N]$$

So $\bigoplus_{m\geq 0}\mathrm{H}^0(X,L^m)$ embeds as a summand of regular ring $K[x,y]$ after a finite extension, and by [TW18, Lemma 3.17] we get that $C(X,L)$ is strongly F-regular, in particular it is $(0,F)$-rational. Conversely, $C(X,L)$ F-rational implies $\mathrm{H}^1(X,\mathcal{O}_X)=0$ by Lemma 6.1.

We now assume $d\geq 2$. From Proposition 2.12, $C(X,L)$ is (k,F)-rational if and only if it is F-rational and there exists an N so that $\mathrm{H}^{j+1}_{\mathfrak{m}}(Z_n\Omega^{[i]}_{C(X,L)})=0$ for $n\geq N, 1\leq i\leq k$ and $j\leq d-i$. By reflexivity, local cohomology vanishes in degrees 0,1 and by Lemma 4.3 we have

$$\mathrm{H}^{j+1}_{\mathfrak{m}}(Z_n\Omega^{[i]}_{C(X,L)})=\mathrm{H}^j(U,Z_n\Omega^i_U)\simeq\bigoplus_m\mathrm{H}^j(X,(\pi_*Z_n\Omega^i_U)_m),\quad j\geq 1$$

so we reduce to showing $\mathrm{H}^j(X,(\pi_*Z_n\Omega^i_U)_m)=0$ for all weights m and $1\leq i\leq k, 1\leq j\leq d-i$. Note also that conditions (i)-(ii) are equivalent to $C(X,L)$ pre-(k,F)-injective, by Corollary 5.7.1. As (k,F)-rationality implies pre-(k,F)-injectivity (see the paragraph after Proposition 2.12), we see that $C(X,L)$ (k,F)-rational implies (i)-(iii).

Conversely, suppose $C(X,L)$ is pre-(k,F)-injective and satisfies (iii). By Lemma 6.1 we get that $C(X,L)$ is F-rational, and by Lemma 6.4 we have $\mathrm{H}^j(X,(\pi_*Z_n\Omega^i_U)_m)=0$ for $m\neq 0$, $n\geq N$ for a fixed N and i,j as above, so since also $\mathrm{H}^j(X,(\pi_*Z_n\Omega^i_U)_0)=0$ by (iii), we have that $C(X,L)$ is (k,F)-rational from Proposition 2.12. □

If we relax inequality $k<d/2$ to $k\leq d-2$, Proposition 6.5 is valid by replacing (k,F)-rationality with pre-(k,F)-rationality: indeed, by Corollary 5.7.1, $C(X,L)$ being pre-(k,F)-injective is equivalent to (i) and (ii), and for $1\leq i\leq k$, $1\leq j\leq d-i$ we have exact sequence

$$\mathrm{H}^j(U,B_n\Omega^i_U)\to\mathrm{H}^j(U,Z_n\Omega^i_U)\xrightarrow{C_n}\mathrm{H}^j(U,\Omega^i_U)$$

with $\mathrm{H}^j(U, B_n\Omega^i_U) = 0$ from Lemma 6.2. Thus C_n is the zero map if and only if $\mathrm{H}^j(U, Z_n\Omega^i_U) = 0$, and so by [KW26, Thm. A] and Lemma 6.1 $C(X, L)$ is pre-(k, F)-rational if and only if (i)-(iii) hold.

With more restrictions on X this time (rather than on L), we can get a cleaner description of the (k, F)-rationality criterion. Specifically, we require Hodge symmetry on X; by Remark 3.3 this need not hold even if X is Mazur–Ogus, but it does if X in addition has a smooth proper algebraic lift to $W(k)$.

Corollary 6.5.1. *Suppose X is smooth, projective and satisfies Hodge symmetry. For an integer $1 \leq k < \frac{d}{2}$ and $L \in \mathrm{Pic}(X)$ ample, the cone $C(X, L)$ is (k, F)-rational if and only if*

(i) *$C(X, L)$ is pre-(k, F)-injective*
(ii) *$\mathrm{H}^j(X, \Omega^i_X) = 0$ for $i \leq k$ and $0 < j \neq i$, $\mathrm{H}^1(X, \Omega^1_X)$ is spanned by $c_1(L)$ and we have isomorphisms induced by the cup product:*

$$\mathrm{H}^{i-1}(X, \Omega^{i-1}_X) \xrightarrow{\cup c_1(L)} \mathrm{H}^i(X, \Omega^i_X), \quad 2 \leq i \leq k$$

Proof. We first make the following claim: If $\mathrm{H}^j(X, (\pi_*\Omega^{i-1}_U)_0) = \mathrm{H}^j(X, (\pi_* B\Omega^{i-1}_U)_0) = 0$ for all $j \geq 1$ then $\mathrm{H}^j(X, (\pi_* B\Omega^i_U)_0) = 0$ and the Cartier map (in weight zero)

$$C_n : \mathrm{H}^j(X, (\pi_* Z_n\Omega^i_U)_0) \to \mathrm{H}^j(X, (\pi_*\Omega^i_U)_0)$$

is an isomorphism for all $j, n \geq 1$. This follows from applying exact π_* and taking weight zero components in the following exact sequences:

$$0 \to B\Omega^{i-1}_U \to F_*\Omega^{i-1}_U \to F_*\Omega^{i-1}_U / B\Omega^{i-1}_U \to 0$$

$$0 \to \Omega^{i-1}_U \xrightarrow{C^{-1}} F_*\Omega^{i-1}_U / B\Omega^{i-1}_U \to B\Omega^i_U \to 0$$

$$0 \to F^{n-1}_* B\Omega^i_U \to B_n\Omega^i_U \to B_{n-1}\Omega^i_U \to 0$$

$$0 \to B_n\Omega^i_U \to Z_n\Omega^i_U \xrightarrow{C_n} \Omega^i_U \to 0$$

The first two exact sequences yield $\mathrm{H}^j(X, (\pi_* B\Omega^i_U)_0) = 0$, the third one iteratively gives $\mathrm{H}^j(X, (\pi_* B_n\Omega^i_U)_0) = 0$ for all $n \geq 2$ and then the fourth one implies C_n is an isomorphism in cohomological degrees $j \geq 1$.

Now suppose $C(X, L)$ is (k, F)-rational. We have the base case

$$\mathrm{H}^j(X, (\pi_*\mathcal{O}_U)_0) \simeq \mathrm{H}^j(X, \mathcal{O}_X) = 0 \ \ (j \geq 1), \quad (\pi_* B\Omega^0_U)_0 = 0$$

Working inductively on i, assume $\mathrm{H}^j(X, (\pi_*\Omega^l_U)_0) = \mathrm{H}^j(X, (\pi_* B\Omega^l_U)_0) = 0$ for all $l \leq i - 1$, $j \geq 1$, then by the claim above we have $\mathrm{H}^j(X, (\pi_* B\Omega^i_U)_0) = 0$ and

$$\mathrm{H}^j(X, (\pi_*\Omega^i_U)_0) \simeq \mathrm{H}^j(X, (\pi_* Z_n\Omega^i_U)_0) = 0, \ \ 1 \leq j \leq d - i, \quad n \gg 0$$

The latter vanishing here follows from Proposition 6.5 (iii). Via exact sequence

$$0 \to \Omega^i_X \to (\pi_*\Omega^i_U)_0 \to \Omega^{i-1}_X \to 0 \tag{46}$$

we have $\mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^1(X, \Omega^1_X)$ surjective and $\mathrm{H}^{j-1}(X, \Omega^{i-1}_X) \simeq \mathrm{H}^j(X, \Omega^i_X)$ for $2 \leq j \leq d - i$. Since $\mathrm{H}^1(X, (\pi_*\Omega^i_U)_0) = 0$ we have a surjection $\mathrm{H}^0(X, \Omega^{i-1}_X) \twoheadrightarrow \mathrm{H}^1(X, \Omega^i_X)$, and by Hodge symmetry $\mathrm{H}^0(X, \Omega^{i-1}_X) = 0$ for $i > 1$ (since $\mathrm{H}^{i-1}(X, \mathcal{O}_X) = 0$). By induction on i we get $\mathrm{H}^j(X, \Omega^i_X) = 0$ for $1 \leq j \leq d - i$ and $j \neq i$. For $j > d - i$, Serre duality and Hodge symmetry give

$$\dim \mathrm{H}^j(X, \Omega^i_X) = \dim \mathrm{H}^{d-i}(X, \Omega^{d-j}_X) = 0$$

Overall $\mathrm{H}^j(X,(\pi_*\Omega^i_U)_0)=0$ for $j\geq 1$; this completes the inductive step and we get (ii) for $i\leq k$.

Conversely assume (i)-(ii), then $\mathrm{H}^j(X,\mathcal{O}_X)=0$ for $j>0$ and (46) yields $\mathrm{H}^j(X,(\pi_*\Omega^i_U)_0)=0$ for $j\geq 1$ and $i\leq k$; for the latter we use Hodge symmetry on X again. The claim then gives $\mathrm{H}^j(X,(\pi_*Z_n\Omega^i_U)_0)=0$ for all $n\geq 1$, $1\leq i\leq k$ and $1\leq j\leq d-i$ so by Proposition 6.5 we get $C(X,L)$ (k,F)-rational. □

If we also impose $\Omega^i_{C(X,L)}$ reflexive for $i\leq k$ we can replace (i) with $C(X,L)$ being (k,F)-injective.

Remark 6.6. *One can compare Corollary* 6.5.1 *with the pre-k-rational cone criterion in characteristic* 0 [SVV23, Cor. 7.7]. *Outside the vanishing conditions on $B\Omega^i_X$, which is a purely positive characteristic phenomenon, the only difference in the criteria stems from (k,F)-rationality (and (k,F)-injectivity) requiring the vanishing of cohomology in negative weights:*

$$\mathrm{H}^j(X,\Omega^i_X\otimes L^m)=0 \qquad \text{for } 1\leq j<d-i, i\leq k, m<0$$

Both require the vanishing of the same cohomology groups $\mathrm{H}^j(X,\Omega^i_X)$ for $j\neq i$ and the same cup-product isomorphisms when $j=i$.

Appendix A. Kuznetsov components of cubic fourfolds

Assume that k is an algebraically closed field of characteristic $p>0$. We use the notation $W:=W(k), K:=W[1/p]$. When X is a smooth proper variety over k, its crystalline cohomology is denoted by $\mathrm{H}^\bullet_{\mathrm{cris}}(X/W)$, and we define $\mathrm{H}^\bullet_{\mathrm{cris}}(X/K):=\mathrm{H}^\bullet_{\mathrm{cris}}(X/W)[1/p]$. All varieties are assumed smooth in this section. Main references for this section are [AB22] and [BP]. Given a k-linear stable ∞-category $\mathcal{C}$ (e.g. the bounded derived category of coherent sheaves on a variety over k), one can define its *topological periodic cyclic homology* as the spectrum $\mathrm{TP}(\mathcal{C}):=\mathrm{THH}(\mathcal{C})^{tS^1}$, where $\mathrm{THH}(\mathcal{C})$ is the topological Hochschild homology of $\mathcal{C}$ which its S^1-action. Here, $(-)^{tS^1}$ denotes the Tate construction by S^1. By taking homotopy groups, we define $\mathrm{TP}_i(\mathcal{C}):=\pi_i(\mathrm{TP}(\mathcal{C}))$ for each integer i. They are W-modules, and the $\mathrm{TP}_i(\mathcal{C})$ are 2-periodic with respect to the index i. Upon inverting p, $\mathrm{TP}(\mathcal{C})$ carries a K-semilinear endomorphism arising from the so-called cyclotomic Frobenius, and we write it as

$$\phi:\mathrm{TP}(\mathcal{C})_K\to\mathrm{TP}(\mathcal{C})_K.$$

This induces a semilinear map $\phi:\mathrm{TP}_i(\mathcal{C})_K\to\mathrm{TP}_i(\mathcal{C})_K$ for each i.

When X is a smooth proper variety over k, we set $\mathrm{TP}(X):=\mathrm{TP}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X))$, and its i -th homotopy group is denoted by $\mathrm{TP}_i(X)$. For each i, the pair $(\mathrm{TP}_i(X),\phi)$ is an F-isocrystal, and we will consider the cases of cubic fourfolds and K3 surfaces below.

Remark A.1. $\mathrm{TP}_i(-)$ *enjoys the following additional properties.*

1. *The crystalline-*TP *spectral sequence degenerates rationally. Moreover, the spectral sequence is compatible with the F-isocrystal structures after appropriate Tate twists. As a result, we have an isomorphism of F-isocrystals*
$$\mathrm{TP}_{2i}(X)_K\simeq\oplus_j\mathrm{H}^{2j}_{\mathrm{cris}}(X/K)(i+j).$$
2. *Given a semiorthogonal decomposition $\mathcal{C}=\langle\mathcal{C}_1,\mathcal{C}_2,..,\mathcal{C}_m\rangle$, the projection functors $\mathcal{C}\to\mathcal{C}_j$ induce decompositions*
$$\mathrm{TP}_i(\mathcal{C})=\mathrm{TP}_i(\mathcal{C}_1)\oplus\mathrm{TP}_i(\mathcal{C}_2)\oplus...\,\mathrm{TP}_i(\mathcal{C}_m).$$

Proposition A.2. *Let Y be a K3 surface over k. Then, we have an isomorphism of F-crystals*

$$\mathrm{H}^0_{\mathrm{cris}}(Y/W) \simeq W, \quad \mathrm{H}^4_{\mathrm{cris}}(Y/W) \simeq W(-2).$$

Moreover, all odd degree crystalline cohomologies vanish, and $\mathrm{H}^2_{\mathrm{cris}}(Y/W)$ is free of rank 22.

Proof. If we disregard the F-crystal structures, this is proved in [Del81, Proposition 1.1]. The F-crystal structures of $\mathrm{H}^0_{\mathrm{cris}}(Y/W)$ and $\mathrm{H}^4_{\mathrm{cris}}(Y/W)$ can be deduced from Poincaré duality. □

Proposition A.3. *Let $X \subset \mathbb{P}^5_k$ be a cubic fourfold. Then, we have an isomorphism of F-crystals*

$$\mathrm{H}^0_{\mathrm{cris}}(X/W) \simeq W, \mathrm{H}^2_{\mathrm{cris}}(X/W) \simeq W(-1), \mathrm{H}^6_{\mathrm{cris}}(X/W) \simeq W(-3), \mathrm{H}^8_{\mathrm{cris}}(X/W) \simeq W(-4).$$

All odd degree crystalline cohomologies vanish, and $\mathrm{H}^4_{\mathrm{cris}}(X/W)$ is free of rank 23.

Proof. Ignoring the F-crystal structure for a moment, we calculate cohomologies as W-modules.

One first computes that the dimension of $\mathrm{H}^i_{\mathrm{dR}}(X/k)$ is equal to $1, 0, 1, 0, 23, 0, 1, 0, 1$ when $i = 0, 1, 2, 3, 4, 5, 6, 7, 8$ respectively. This can be done using the degeneration of the Hodge-to-de Rham spectral sequence as in [Huy23, §1.1.6].

Since the odd-degree de Rham cohomologies vanish, the "universal coefficient theorem" for crystalline cohomology shows that odd degree crystalline cohomologies vanish as well as that even degree crystalline cohomologies are torsion-free. The same theorem then implies that $\mathrm{H}^*_{\mathrm{dR}}(X/k) \simeq \mathrm{H}^*_{\mathrm{cris}}(X/W) \otimes_W k$, giving us the ranks. The F-crystal structure on $\mathrm{H}^0_{\mathrm{cris}}(X/W)$ is obvious, so the F-crystal structure on $\mathrm{H}^8_{\mathrm{cris}}(X/W)$ follows by Poincaré duality. The image of an ample line bundle L in $\mathrm{H}^2_{\mathrm{cris}}(X/W)$ is nonzero and must satisfy $pc_1(L) = \sigma(c_1(L))$ where $F : X \to X$ is the Frobenius map, and σ is the Frobenius action on $\mathrm{H}^2_{\mathrm{cris}}(X/W)$. It follows that $\mathrm{H}^2_{\mathrm{cris}}(X/W) \simeq W(-1)$ by the Dieudonné–Manin classification. We then compute $\mathrm{H}^6_{\mathrm{cris}}(X/W)$ by Poincaré duality. □

A byproduct of the proofs of the two lemmas above is the following (cf. Example 3.2):

Proposition A.4. *K3 surfaces and cubic fourfolds are Mazur–Ogus.*

Let X be a cubic fourfold. Define the *Kuznetsov component* $\mathrm{Ku}(X)$ of X as the right orthogonal complement of the exceptional collection $\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, so that we have a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathrm{Ku}(X), \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle.$$

By Remark A.1, we observe:

Proposition A.5. *Let X be a cubic fourfold over k. Then, we have an isomorphism of F-isocrystals*

$$\mathrm{TP}_{-2}(X)_K \simeq \mathrm{TP}_{-2}(\mathrm{Ku}(X))_K \oplus K(-1)^{\oplus 3}.$$

Proposition A.6. *Let X be a cubic fourfold, and let Y be a K3 surface with $\mathrm{Ku}(X) \simeq \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$. Then, we have an isomorphism of F-isocrystals*

$$\mathrm{H}^4_{\mathrm{cris}}(X/K)(1) \simeq \mathrm{H}^2_{\mathrm{cris}}(Y/K) \oplus K(-1)$$

Proof. By Proposition A.5,

$$\mathrm{TP}_{-2}(X)_K \simeq \mathrm{TP}_{-2}(Y)_K \oplus K(-1)^{\oplus 3}.$$

TP of X and Y can be written in terms of crystalline cohomologies by Remark A.1, which can be simplified by Propositions A.2, A.3. Substituting these expressions into the isomorphism above gives the result. □

Corollary A.6.1. *In the situation of Proposition* A.6*, X is Newton–Hodge ordinary in degree 4 if and only if Y is Newton–Hodge ordinary in degree 2.*

Proof. In view of the Hodge numbers of X, the former happens when the slope-i part of $\mathrm{H}^4_{\mathrm{cris}}(X/K)$ has dimensions $0, 1, 21, 1, 0$ when $i = 0, 1, 2, 3, 4$, respectively. By the proposition, this is equivalent to the condition that the slope-i part of $\mathrm{H}^2_{\mathrm{cris}}(Y/K)$ has dimensions $1, 20, 1$ when $i = 0, 1, 2$ respectively. This is equivalent to the Newton–Hodge ordinarity in degree 2 for Y. □

Remark A.7. For a K3 surface X, Newton-Hodge ordinarity in degree 2 is equivalent to Newton–Hodge 1-ordinarity in degree 2. To see this, we assume X is Newton–Hodge 1-ordinary. Recall that the Hodge polygon of K3 surfaces in degree 2 is piecewise linear with vertices $(0, 0), (1, 0), (21, 20), (22, 22)$. By assumption, the slope 0 subspace of $\mathrm{H}^2_{\mathrm{cris}}(X/K)$ has dimension 1. Poincaré duality pairs $\mathrm{H}^2_{\mathrm{cris}}(X/K)$ with itself, so the slope 2 subspace also has dimension 1. This completely determines the Newton polygon, and it coincides with the Hodge polygon, so we have full ordinarity.

A similar argument (cf. Lemma 3.9) shows that for a cubic fourfold, Newton–Hodge ordinarity in degree 4 is equivalent to Newton–Hodge 2-ordinarity in degree 4.

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